VARIATIONAL ANALYSIS OF PREFERENCE RELATIONS AND UTILITY

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Overview of Preferences and Utility

Vectors of goods: $x = (x_1, \ldots, x_g),\ y = (y_1, \ldots, y_g)$: in $\mathbb{R}^g_{++}$

Preferences: $x \preceq y$ ($x \sim y$ if also $y \preceq x$, otherwise $x \prec y$)

Initial axioms used by economists

(A1) completeness: $\forall x, y$, either $x \preceq y$ or $y \preceq x$

(A2) transitivity: $x \preceq y,\ y \preceq z \implies x \preceq z$

(A3) closedness: $R := \{(x, y) \mid x \preceq y\}$ is closed in $\mathbb{R}^g_{++} \times \mathbb{R}^g_{++}$

(A4) monotonicity: $x \prec y$ if $x \leq y$, $x \neq y$

Utility representation: $u : \mathbb{R}^g_{++} \to \mathbb{R}$ with $u(x) \leq u(y) \Leftrightarrow x \preceq y$

- (A1)+(A2) $\Rightarrow$ existence, (A3) $\Rightarrow$ continuous
- (A4) $\Rightarrow$ strictly increasing in all variables

Challenge: securing also properties of <u>smoothness</u> and <u>concavity</u>

Quasi-Concavity and Quasi-Smoothness

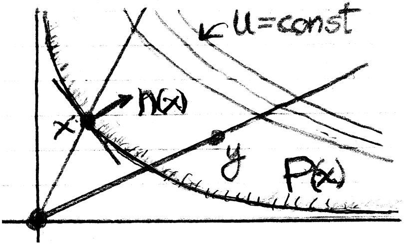
Further axioms used by economists

(A5) the sets $P(x) = \{y \,|\, x \preceq y\}$ are convex

$P : \boldsymbol{R}^g_{++} \rightrightarrows \boldsymbol{R}^g_{++}$ is convex-set-valued, $\mathrm{gph}\, P = R$

(A6) only one hyperplane supports $P(x)$ at x

$N_{P(x)}(x) = \{-\lambda n(x) \,|\, \lambda \geq 0\}$ for unique "unit" $n(x)$



$\mathrm{int}\, P(x) = \{y \,|\, x \prec y\}$

$P(x) = P(y) \iff x \sim y$

Basic utility: for any $e \in \boldsymbol{R}^g_{++}$, $u_e(x) :=$ unique t with $te \sim x$

The Key to Smooth Utility Representations

Scaling structure: $\Theta_{x,y} = \{(r,s) \mid rx \sim sy\}$ for any $x, y \in R^g_{++}$

$\Theta_{x,y} = \operatorname{gph} \theta_{x,y}$ for a function $\theta_{x,y}$ from $(0,\infty)$ onto $(0,\infty)$

$\theta_{x,y}$ is one-to-one, continuous, increasing, and intrinsic

Theorem: first-order smoothness, "$\mathcal{C}^1$ preferences"

$\exists$ utility function $u \in \mathcal{C}^1$ with $\nabla u(x) \neq 0$ always

$\iff$ the basic utility functions u_e have this character

$\iff$ the scaling functions $\theta_{x,y}$ have derivatives $\theta'_{x,y}(r)$ that depend continuously on x, y

$$\nabla u_e(x) = [\theta'_{e,x}(1)/n(x) \cdot x]\, n(x)$$

General utility from basic utility: for any $e \in R^g_{++}$,

$u(x)$ represents the given preference relation

$\iff$ $u(x) = \theta(u_e(x))$ for an increasing function θ

Characterizing Higher-Order Smoothness

Recall: $-n(x)$ is the unit normal to $P(x)$ at x

Quasi-smoothness, first-order: the mapping $x \mapsto n(x)$ is $\mathcal{C}^0$

(this follows from facts about convergencing convex sets)

$\Longrightarrow$ the boundary of $P(x)$ is a $\mathcal{C}^1$ manifold

Quasi-smoothness, second-order: the mapping $x \mapsto n(x)$ is $\mathcal{C}^1$

$\Longrightarrow$ the boundary of $P(x)$ is a $\mathcal{C}^2$ manifold

Theorem: second-order smoothness, "$\mathcal{C}^2$ preferences"

$\exists$ utility function $u \in \mathcal{C}^2$ with $\nabla u(x) \neq 0$ always

$\Longleftrightarrow$ the basic utility functions u_e have this character

$\Longleftrightarrow$ (a) second-order quasi-smoothness holds, and

(b) the scaling functions $\theta_{x,y}$ have derivatives $\theta'_{x,y}(r)$ that depend continuously differentiably on x, y

(this can proceed to higher and higher orders)

Graphical Smoothness of Preferences

Recall: $R = \operatorname{gph} P = \{(x,y) \,|\, x \preceq y\}$ closed set $\subset \mathbf{R}^g_{++} \times \mathbf{R}^g_{++}$

$\operatorname{bdry} R = \{(x,y) \,|\, x \sim y\}$, $\operatorname{int} R = \{(x,y) \,|\, x \prec y\}$

R is epi-Lipschitzian, hence $\operatorname{bdry} R$ is a Lipschizian manifold

Definition of graphical smoothness:

first-order: $\operatorname{bdry} R$ is a $\mathcal{C}^1$ manifold

second-order: $\operatorname{bdry} R$ is a $\mathcal{C}^2$ manifold

Theorem: utility smoothness versus graphical smoothness

The previous smoothness properties correspond to these properties

plus: the mapping $P : x \mapsto P(x)$ is sub-Lipschitz continuous

$$\forall \rho,\ \exists \kappa: \quad P(x') \cap \rho \mathbf{B} \subset P(x) + \kappa |x' - x| \mathbf{B} \quad \text{for } x, x' \in \rho \mathbf{B}$$

Existence of Concave Utility Representations

Strong convexity of preferences: sets $P(x)$ "reliably curved"

locally around any $\bar{x}$, there exists $\sigma > 0$ such that

$$x, x' \sim \bar{x} \implies [n(x') - n(x)]\cdot[x' - x] \leq -\sigma|x' - x|^2$$

Theorem: guaranteed concavity of utility

For $\mathcal{C}^2$ preferences and any compact convex set $C \subset \boldsymbol{R}^g_{++}$,
$\exists$ representation by a $\mathcal{C}^2$ utility function $u(x)$ that is
concave relative to C and moreover minimally so

Minimally concave utility representation: $u(x)$ on C such that
all others are of the form $\theta(u(x))$ for a concave function θ

minimally concave utility is unique up to affine rescaling!

thus unique up to a choice of units, like Celsius/Fahrenheit

References

[1] R.. T. Rockafellar (**2022**), "Variational analysis of preference relations and their utility representation," *Pure and Applied Functional Analysis*, accepted.
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[2] J. Deride, A. Jofré, R. T. Rockafellar (**2023**), "Reaching an equilibrium of prices and holdings through direct buying and selling," *Economic Theory*, submitted.

downloads: sites.math.washington.edu/∼rtr/mypage.html