CONSTRUCTION OF OPERATORS WITH WILD DYNAMICS.

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Non linear phenomena with only linear operators.

Wild dynamics with 3 orthogonal projections.

Let H be an infinite-dimensional Hilbert space.

Theorem (Kopecka-Muller-Paszkiewicz). There exist three orthogonal projections P_1, P_2, P_3 onto closed subspaces of H such that for every $z_0 \in H\backslash\{0\}$, there exist k_1, $k_2, \cdots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $z_{n+1} = P_{k_n} z_n$ does not converge in norm.

Hypercyclic operators.

Let $(X, \|\cdot\|_X)$ be a Banach space and $T \in \mathcal{L}(X)$.

- Set of hypercyclic vectors of T :
  $HC(T) := \{x \in X;$ the sequence $(T^n x)$ is dense in $X\}$

  T is hypercyclic if $HC(T) \neq \emptyset$. In this case, $HC(T)$ is dense.

  Theorem : For every separable Banach space X such that $dim(X) = \infty$, there exists $T \in \mathcal{L}(X)$ such that T is hypercyclic.

  Moreover, we can construct T so that $I - T$ is compact.

  Read : There exists $T \in \mathcal{L}(\ell^1(\mathbb{N}))$ such that $HC(T) = \ell^1(\mathbb{N}) \backslash \{0\}$.

Universality of hypercyclic operators.

Theorem (Feldmann) :
There exists a separable Hilbert space H
and there exists an hypercyclic operator $T \in \mathcal{L}(H)$
with the following property :

for every compact metric space K,
for every continuous function $f : K \to K$,

there exists a compact subset L of H stable by T and an homeomorphism $\Phi : K \to L$ such that

$$\Phi \circ f = T \circ \Phi.$$

A theorem of Hajek, Smith and Augé.

- $U(T) = \{x \in X$; the sequence $(\|T^n x\|)$ is unbounded.

  Uniform boundedness principle. $U(T)$ is either empty or residual.

- $A(T) = \{x \in X$; the sequence $(\|T^n x\|)$ tends to $+\infty\}$.

  Is $A(T)$ either empty or dense ?

- Answer : no. Hajek and Smith constructed counterexamples in every separable Banach space with symmetric basis.

- (Muller) If X is real and $\sum_{n=1}^{+\infty} \frac{1}{\|T^n\|} < +\infty$ then $A(T)$ is dense

A theorem of Hajek, Smith and Augé.

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- $A(T) = \{x \in X;$ the sequence $(\|T^n x\|)$ tends to $+\infty\}$.
- $R(T) = \{x \in X;\ \liminf \|T^n x - x\| = 0\}$ (recurrent points of T).

Theorem (J. M. Augé) : For every separable Banach space X with $dim(X) = \infty$, there exists $T \in \mathcal{L}(X)$ such that

- $\Big\{R(T), A(T)\Big\}$ is a partition of X,

- both $R(T)$ and $A(T)$ have non empty interior.

Moreover, we can construct T so that $I - T$ is compact.

Recurrent points.

- $A(T) = \{x \in X;$ the sequence $(\|T^n x\|)$ tends to $+\infty\}$.
- $B(T) = \{x \in X;\quad 0 < \liminf \|T^n x - x\| < +\infty\}$
- $R(T) = \{x \in X;\ \liminf \|T^n x - x\| = 0\}$ (recurrent points of T).

Theorem (Augé) : For every separable Banach space X with $dim(X) = \infty$, there exists $T \in \mathcal{L}(X)$ such that

- $\Big\{R(T), A(T)\Big\}$ is a partition of X,
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Theorem (Deville - Strouse) : For every separable Banach space X with $dim(X) = \infty$, there exists $T \in \mathcal{L}(X)$ such that

- $\Big\{R(T), B(T)\Big\}$ is a partition of X,
- both $R(T)$ and $B(T)$ have non empty interior.

The sum of two operators.

Let D, R be two operators on X satifying :

$$R \circ D = R \qquad R^2 = 0$$

Denote $T = D + R$. Then for each $n \geq 1$,

$$T^n = D^n + (I + D + \cdots + D^{n-1}) \circ R.$$

Example.

$D, R \in \mathcal{L}(\mathbb{C}^2)$, D with matrix $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ where $\lambda = e^{i\pi/m}$,

R with matrix $\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$

The matrix of $\quad T = D + R \quad$ is $\begin{pmatrix} 1 & r \\ 0 & \lambda \end{pmatrix}$

The operator T on $\mathbb{C}^2$.

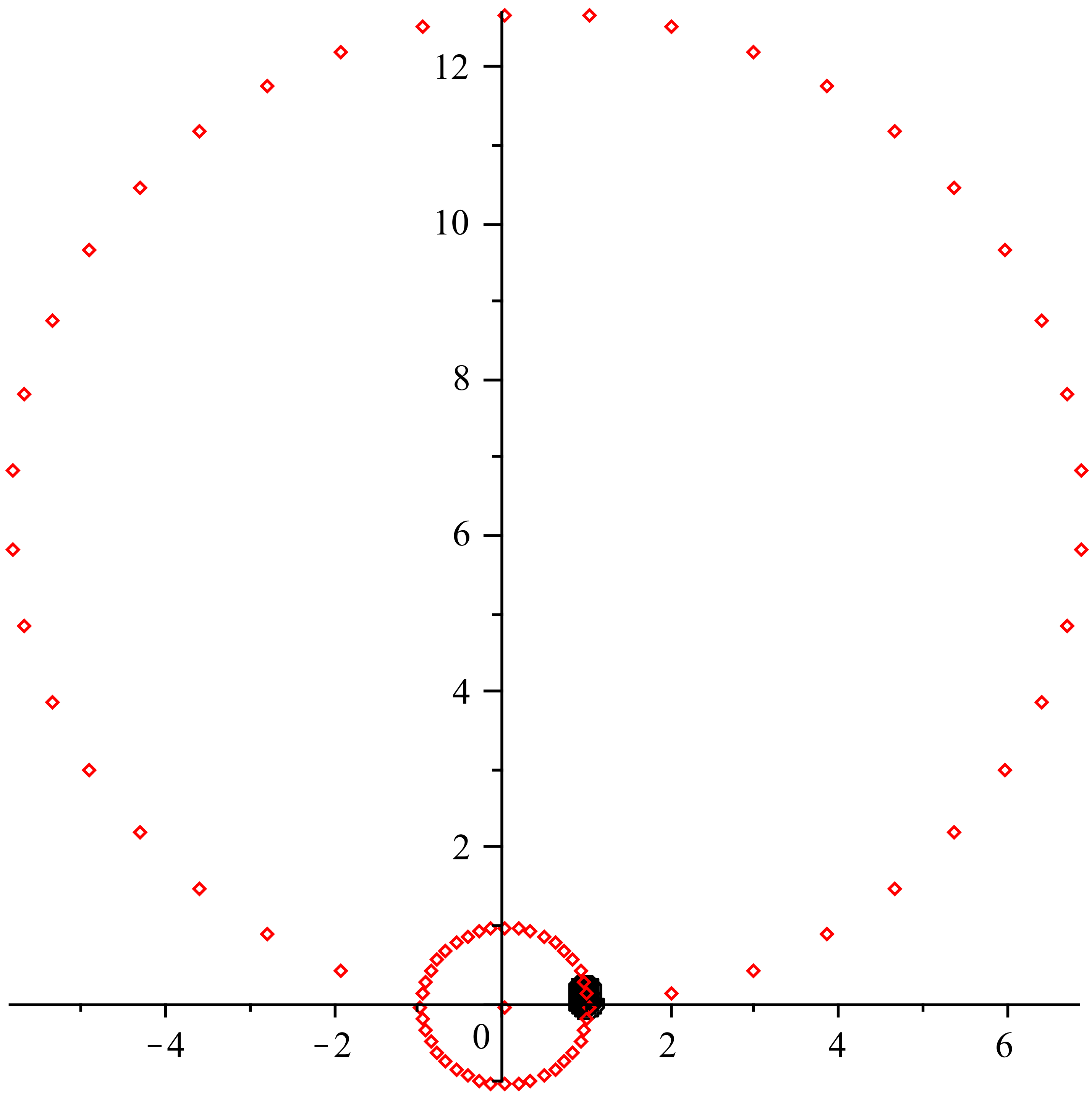
$T \in \mathcal{L}(\mathbb{C}^2)$ with matrix $\begin{pmatrix} 1 & r \\ 0 & \lambda \end{pmatrix}$

Matrix of T^n $\quad \begin{pmatrix} 1 & r\lambda_n \\ 0 & \lambda^n \end{pmatrix}$ where $\quad \lambda_n = \sum_{k=0}^{n-1} \lambda^k.$

If $\lambda = e^{i\pi/m}$ and $\frac{1}{m} \ll r \ll 1$, then

- $\|T - Id\|$ is small
- $T^{2m} = Id$
- $\|T^m\|$ is of order rm (hence large),

because $\quad \frac{2}{\pi} n \le |\lambda_n| \le n \quad$ if $n \le m$.



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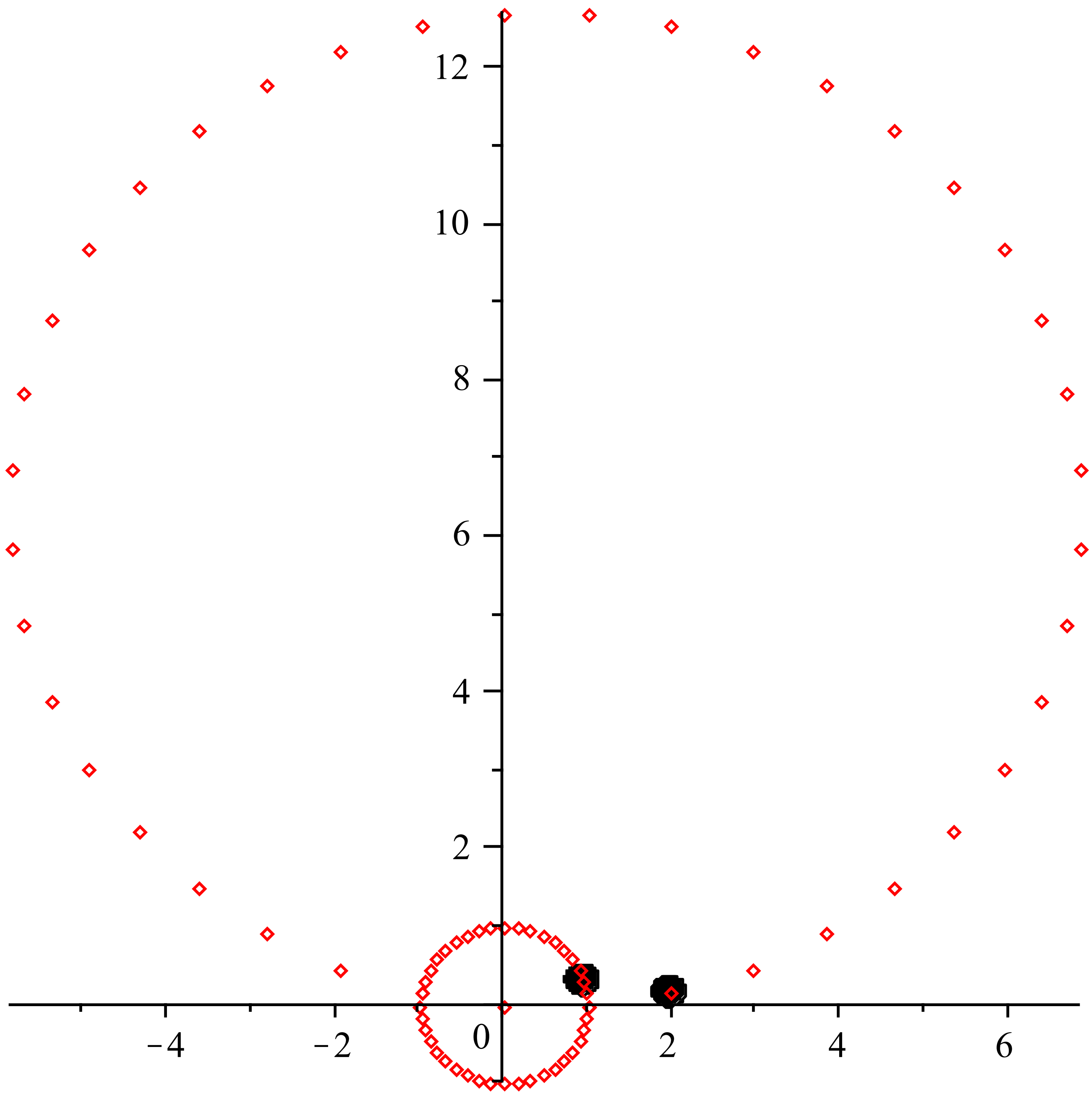
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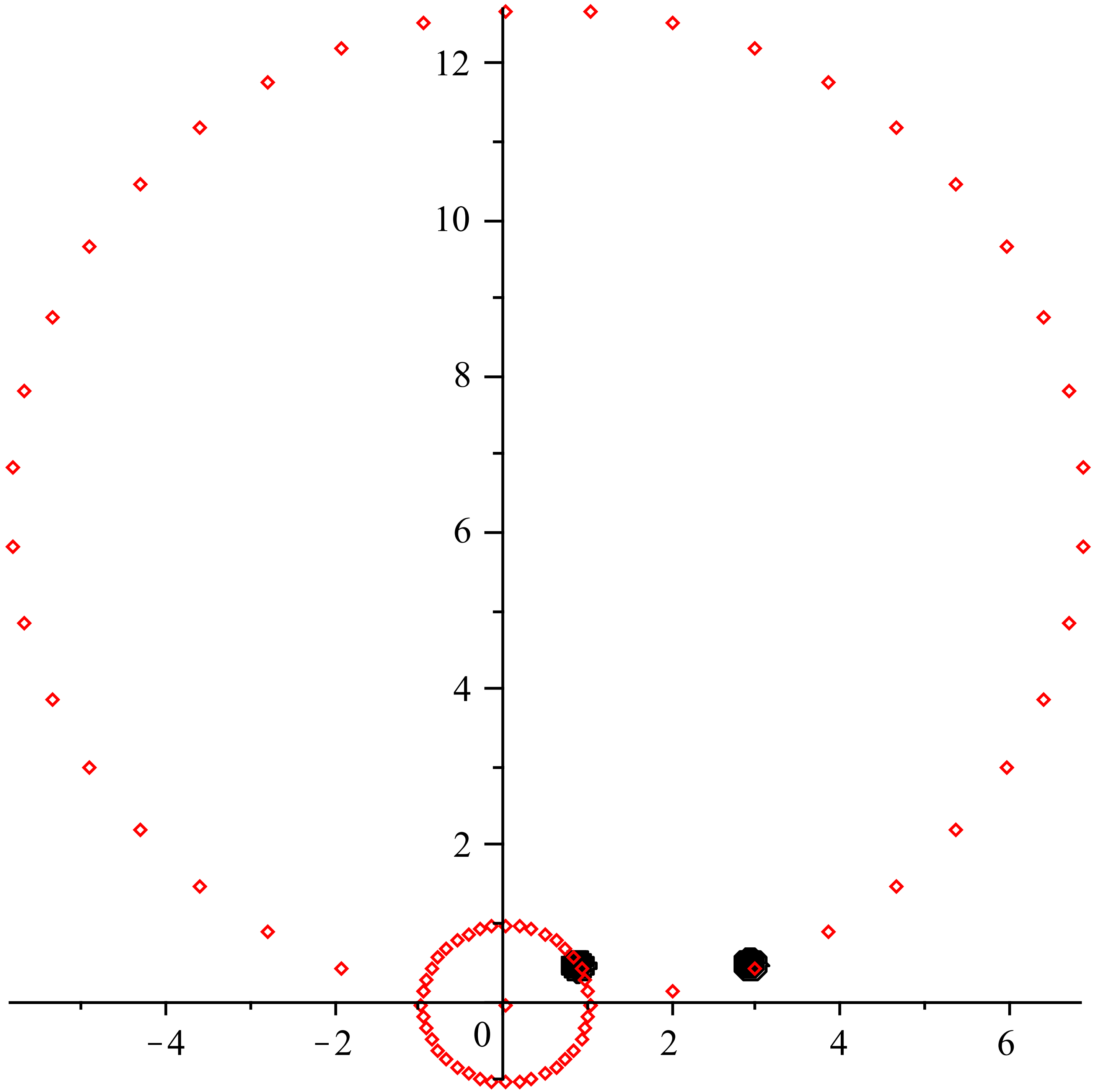

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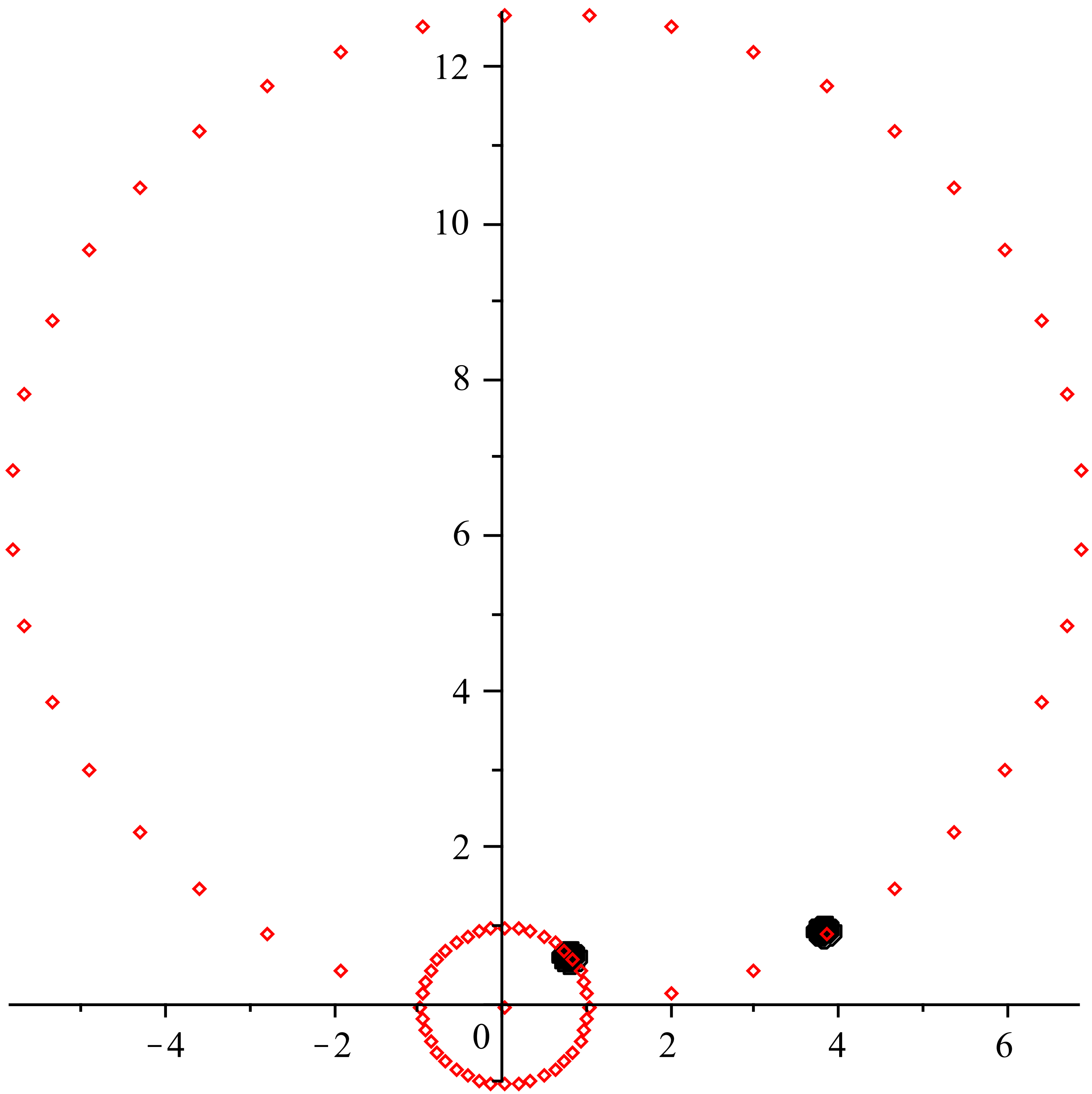

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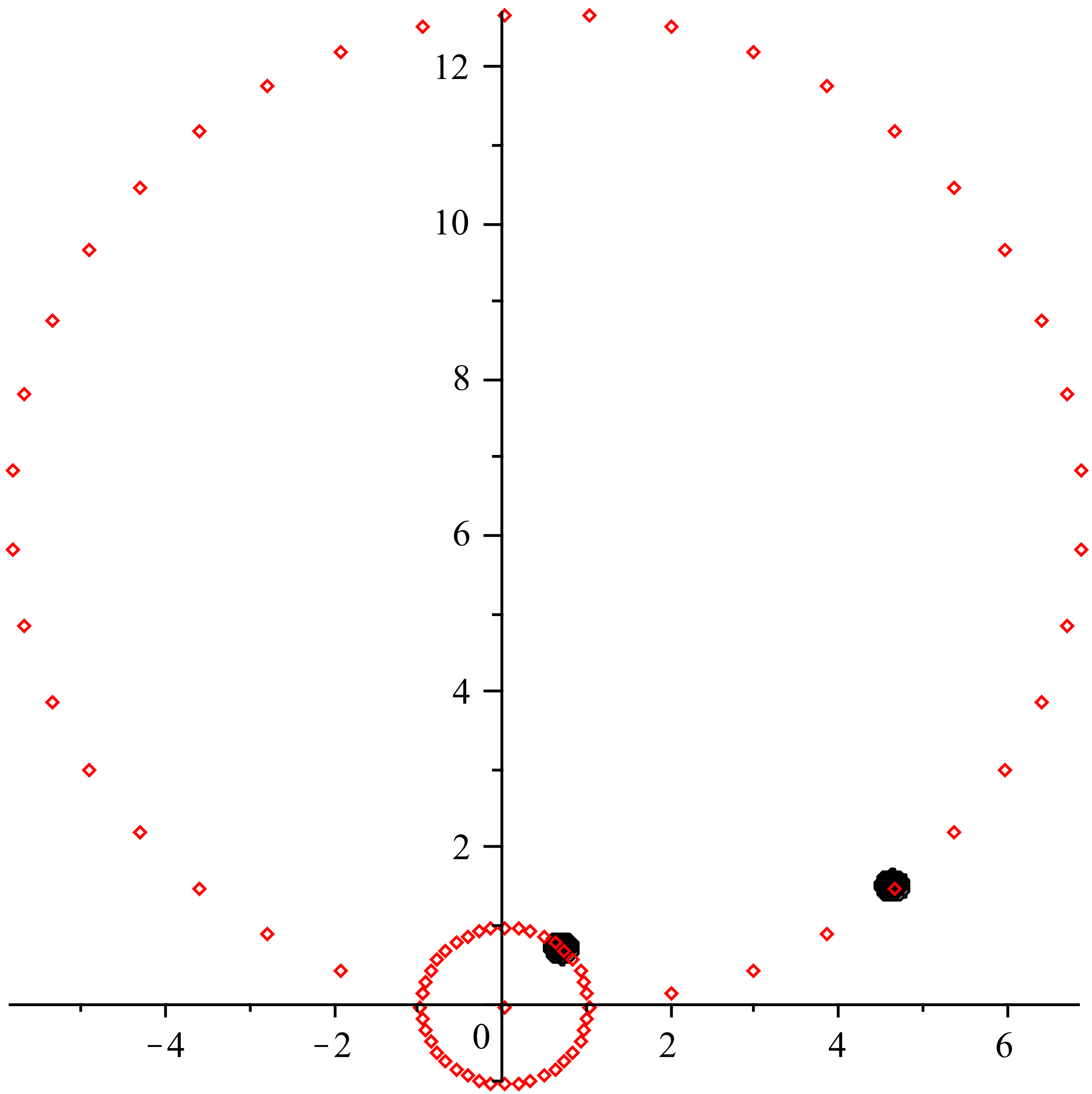

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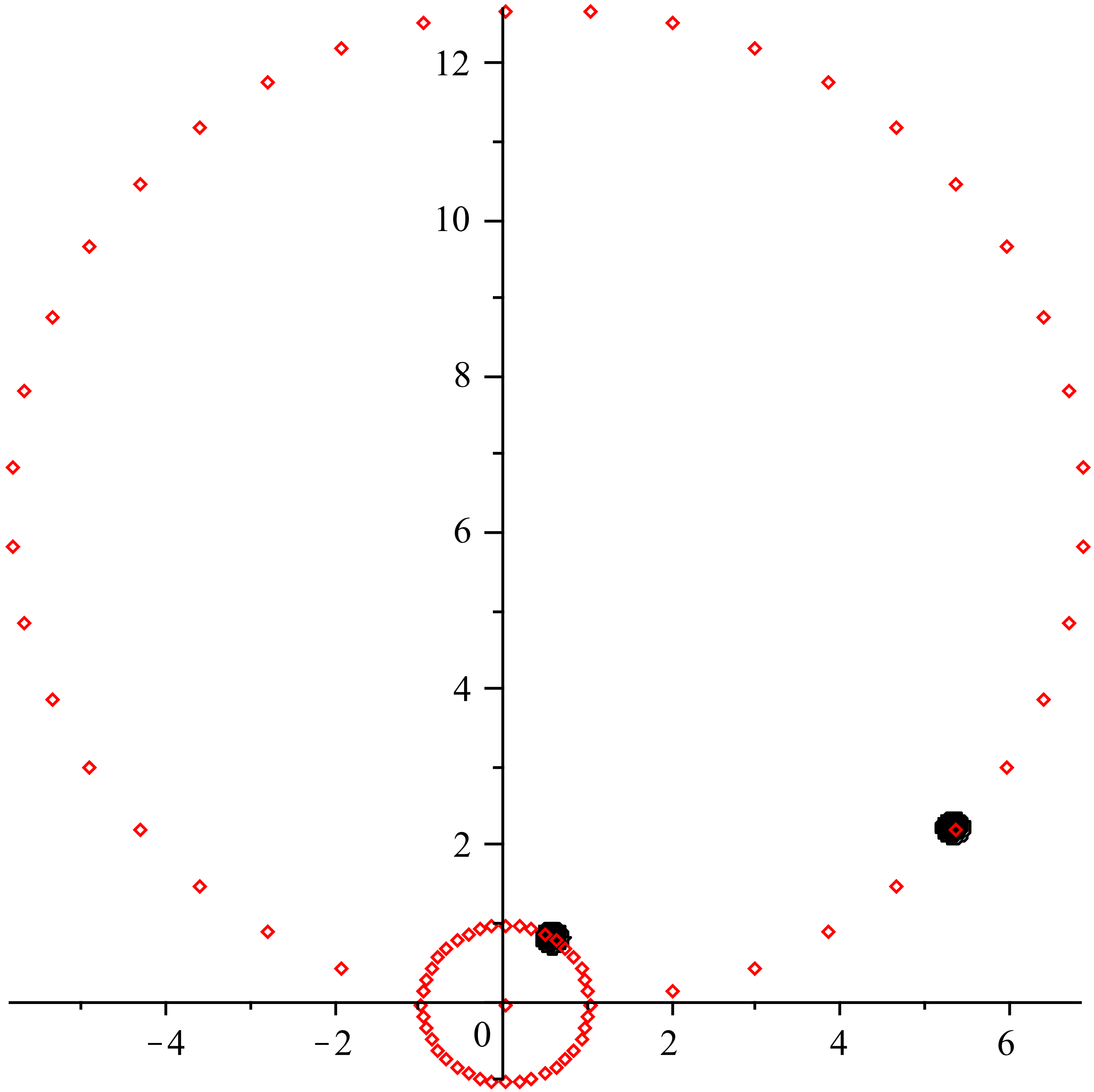
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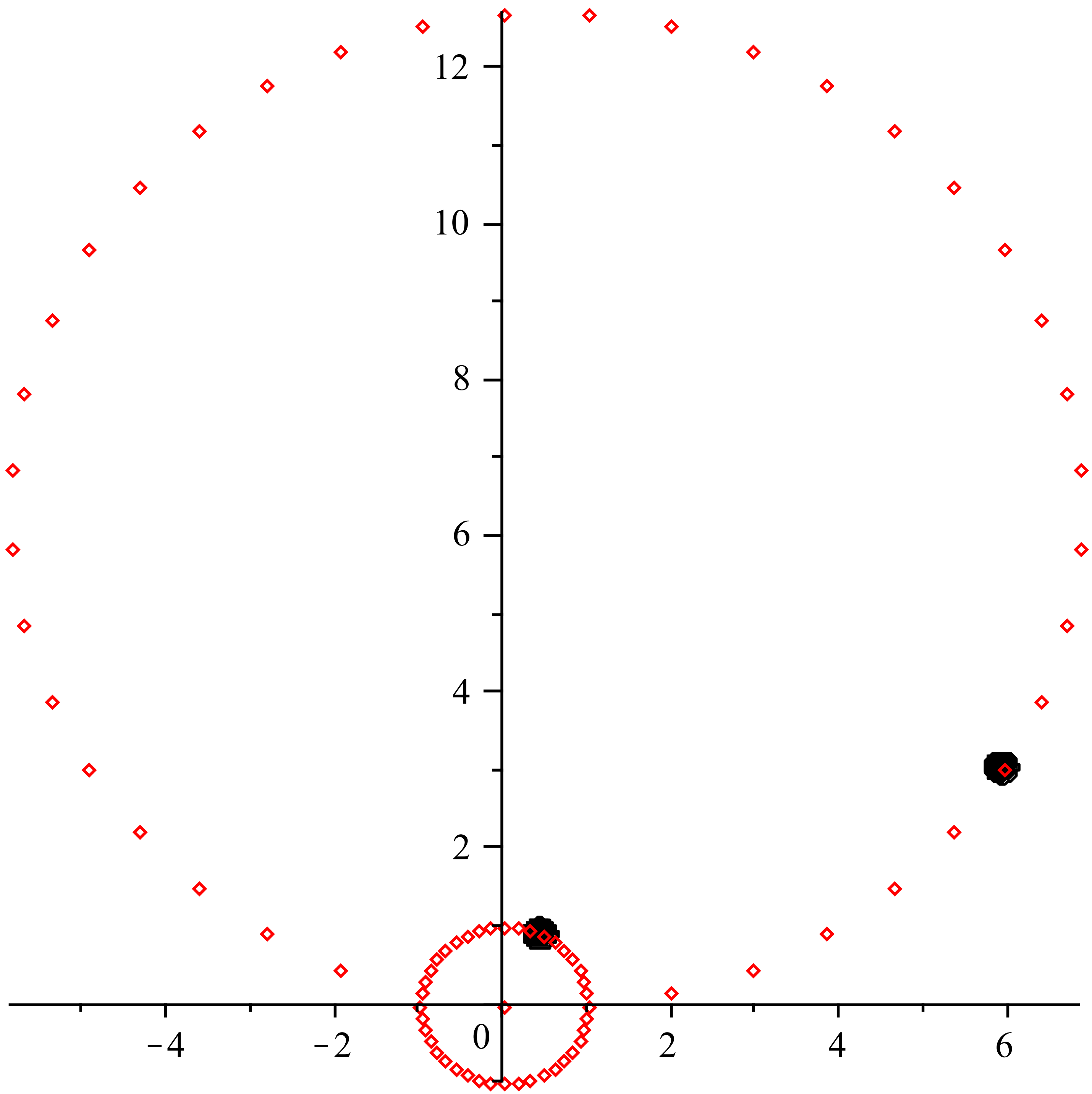

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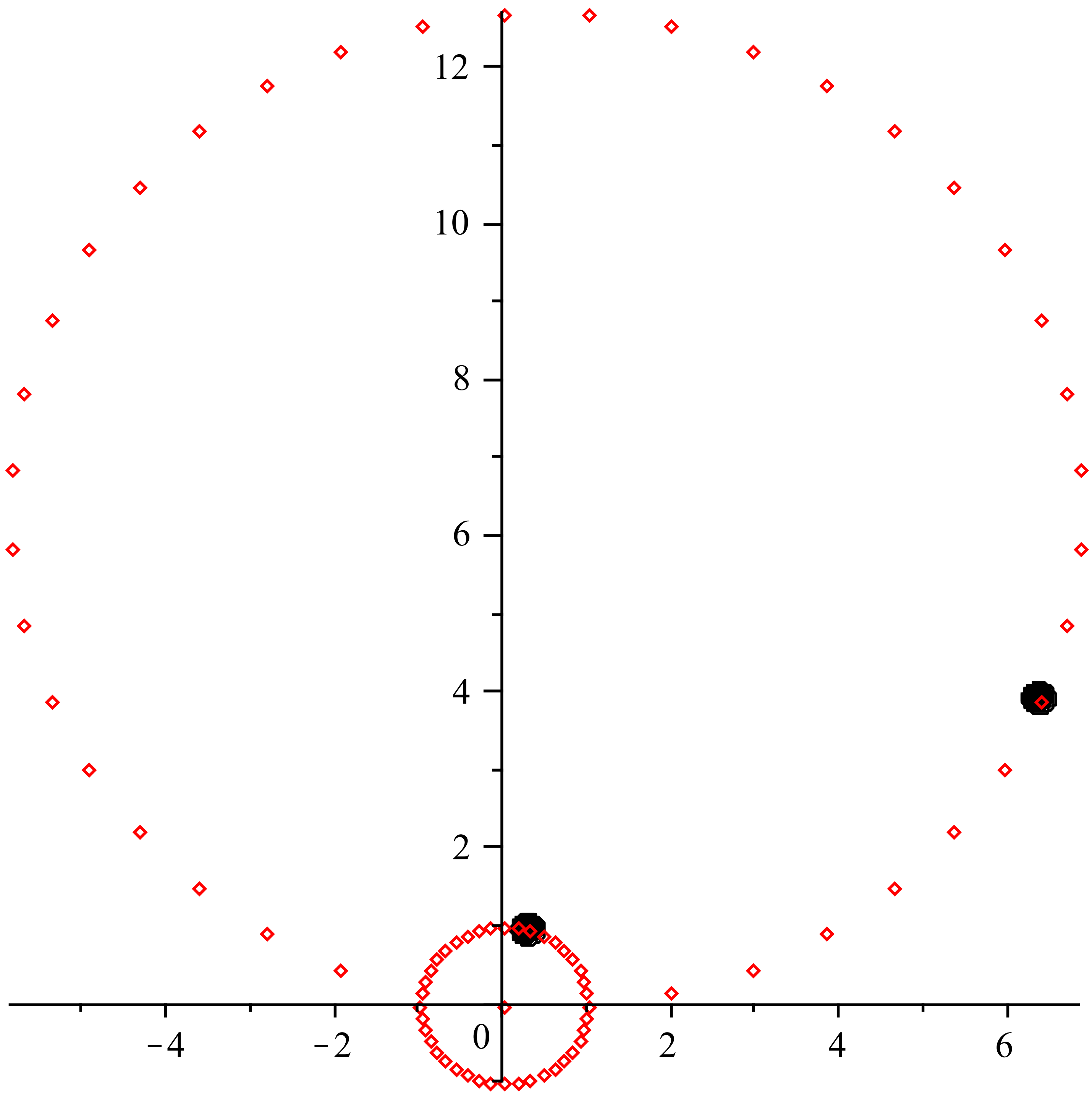
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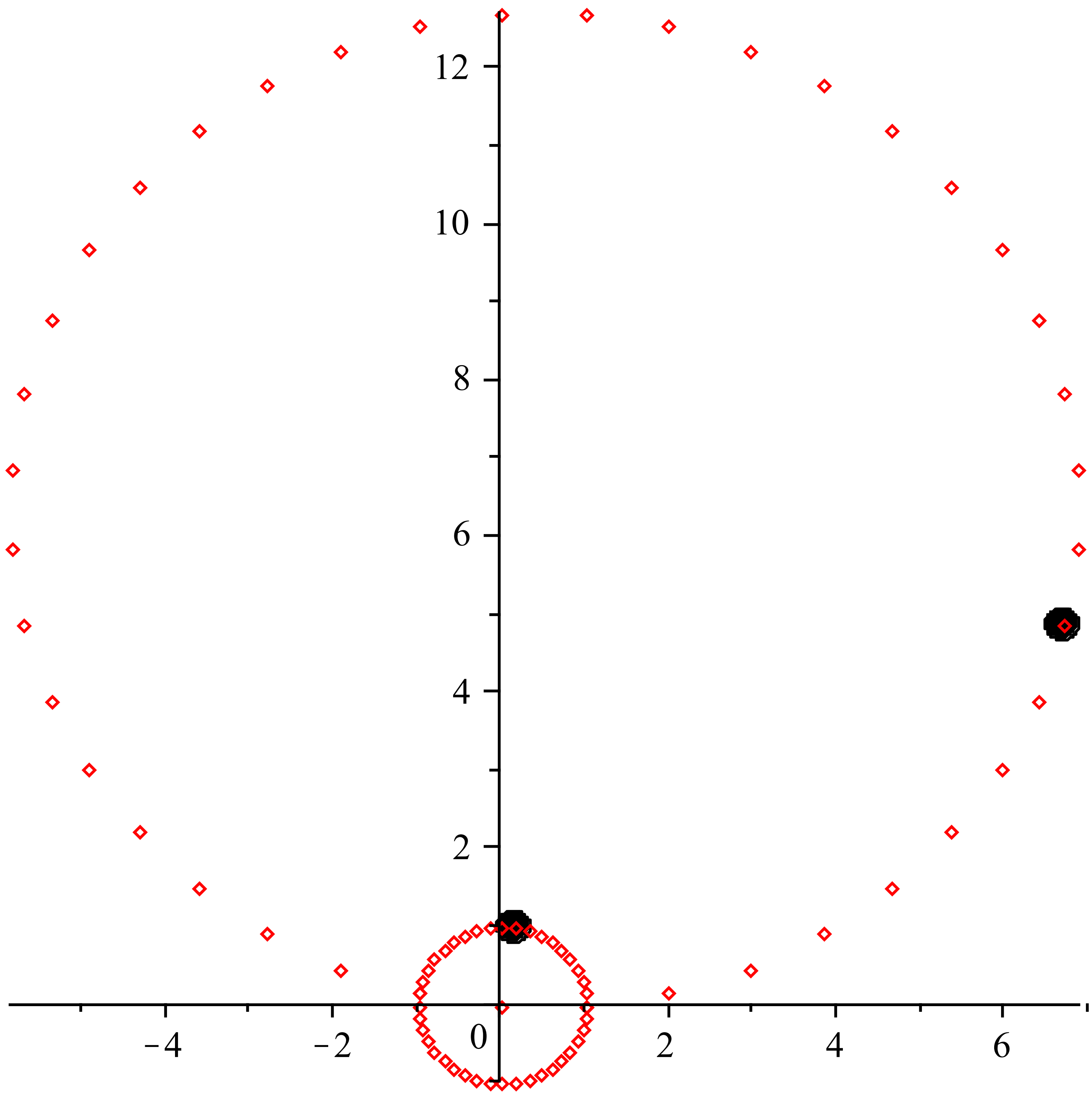

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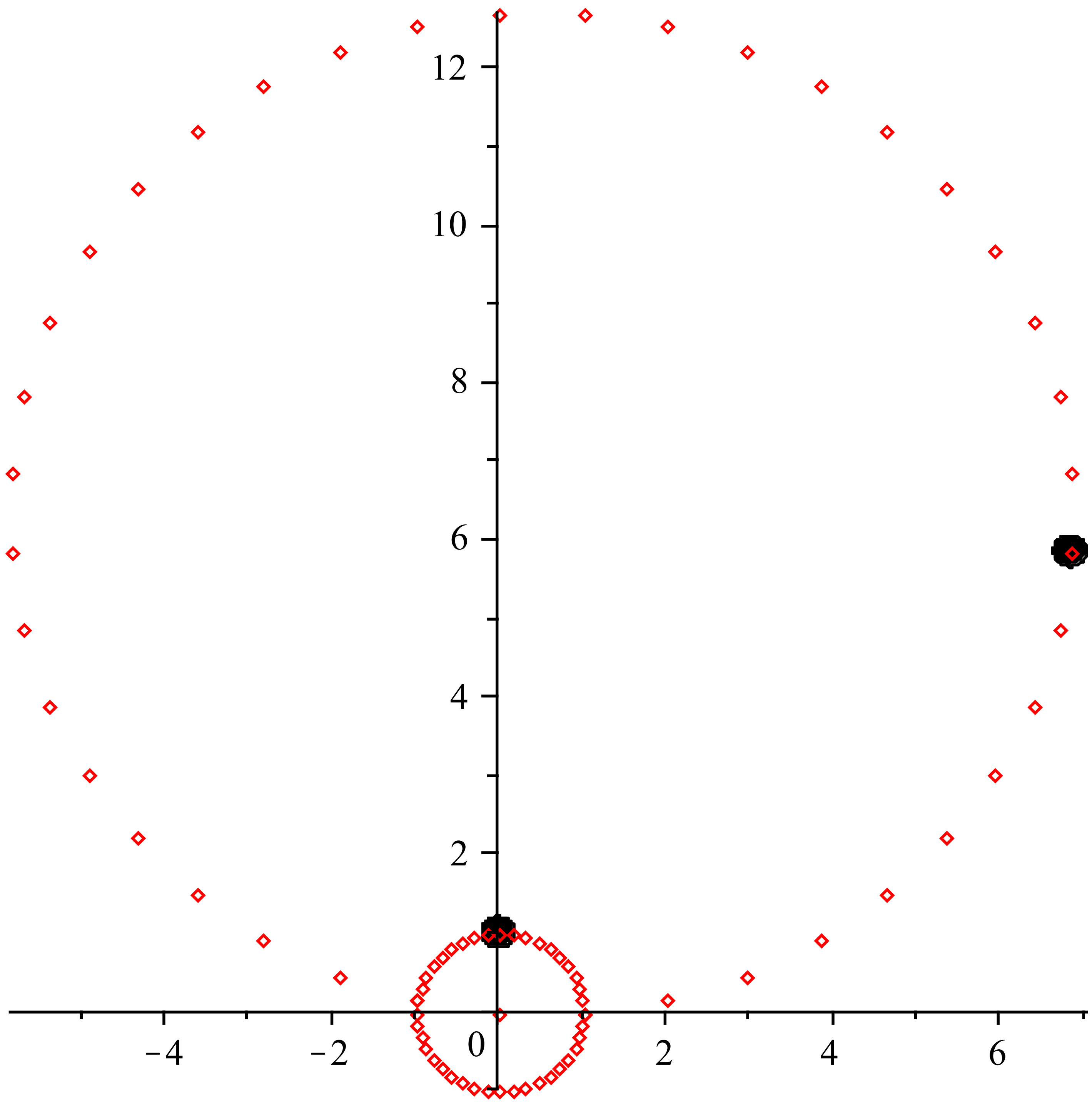

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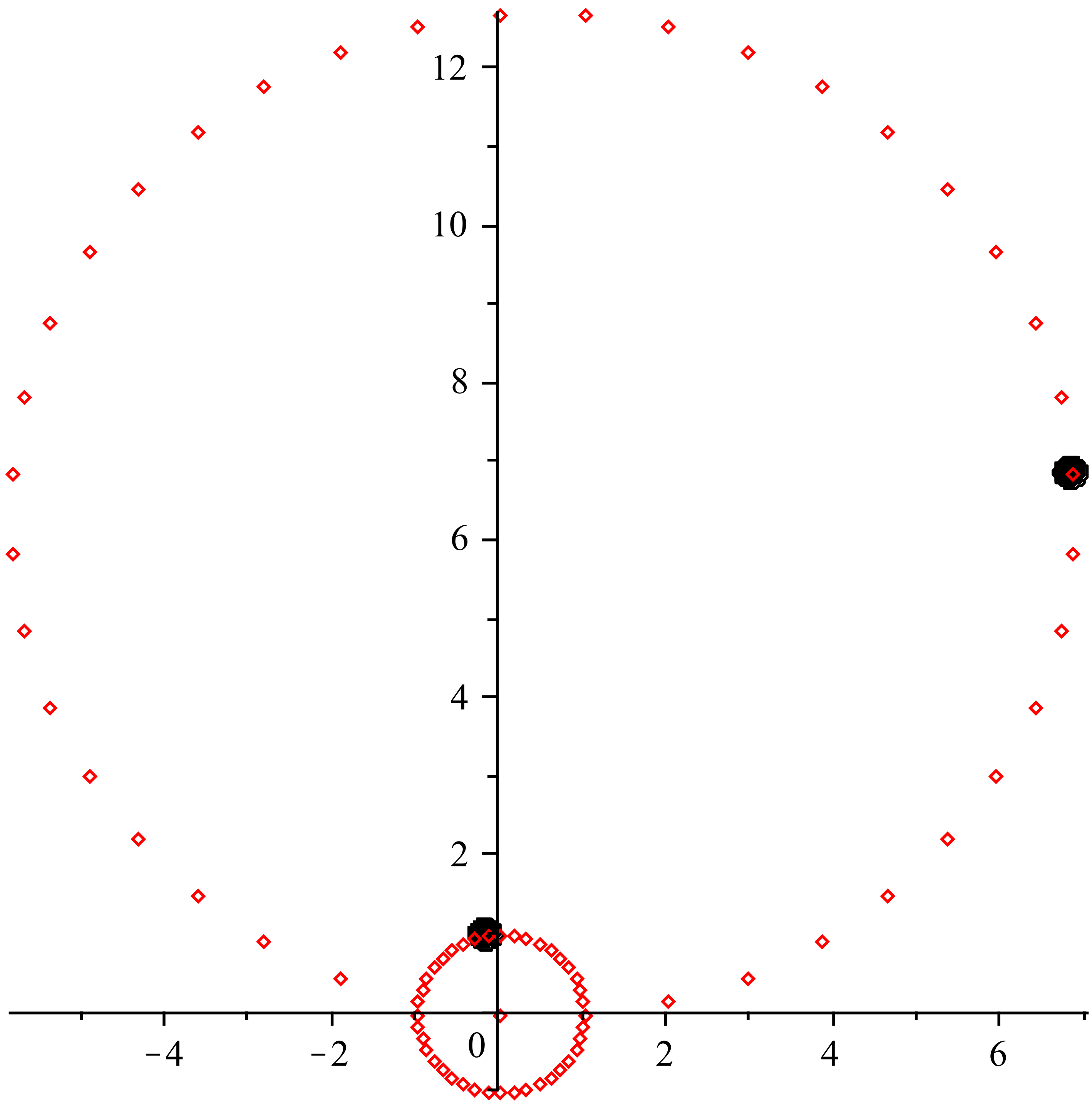
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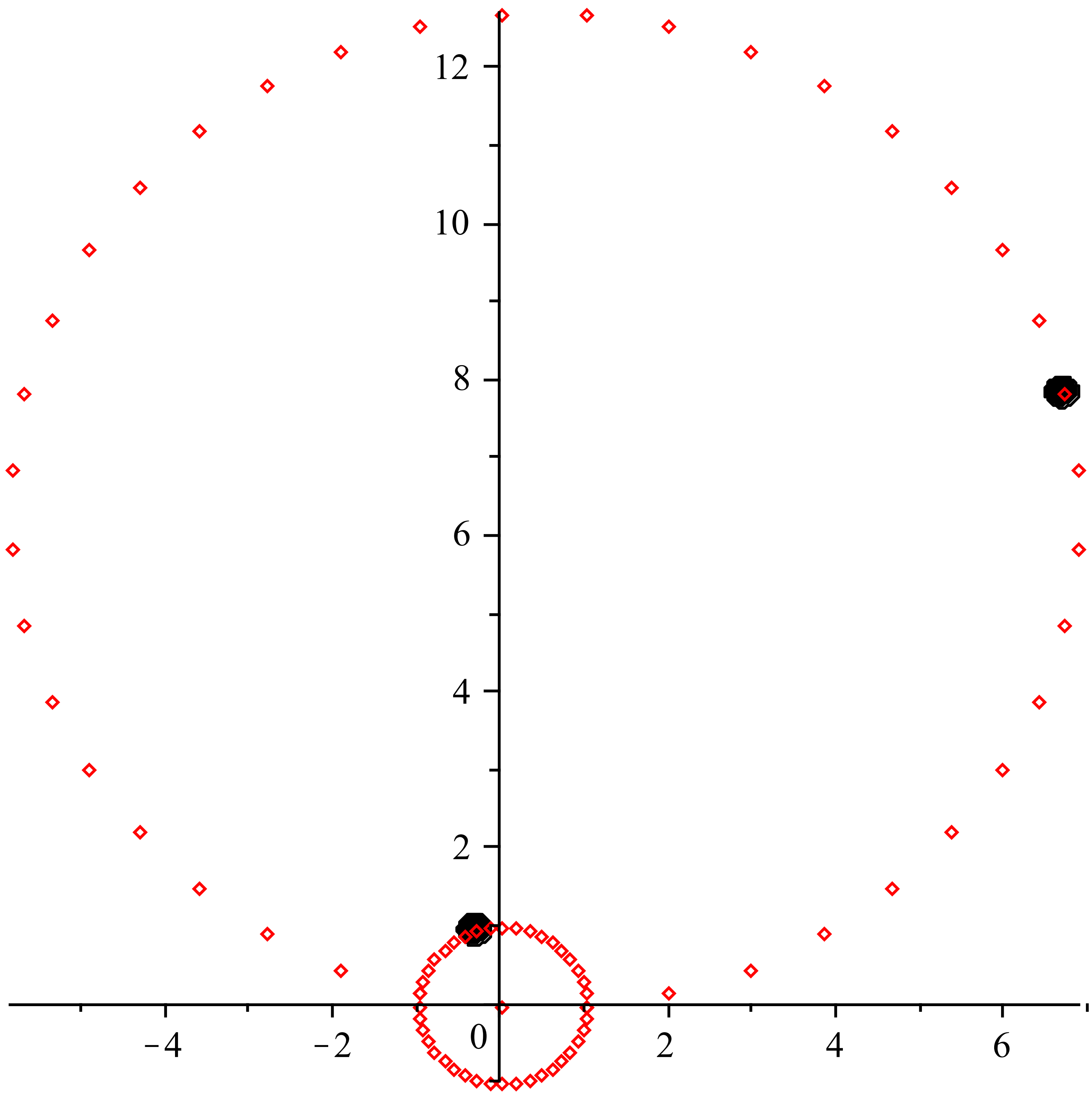

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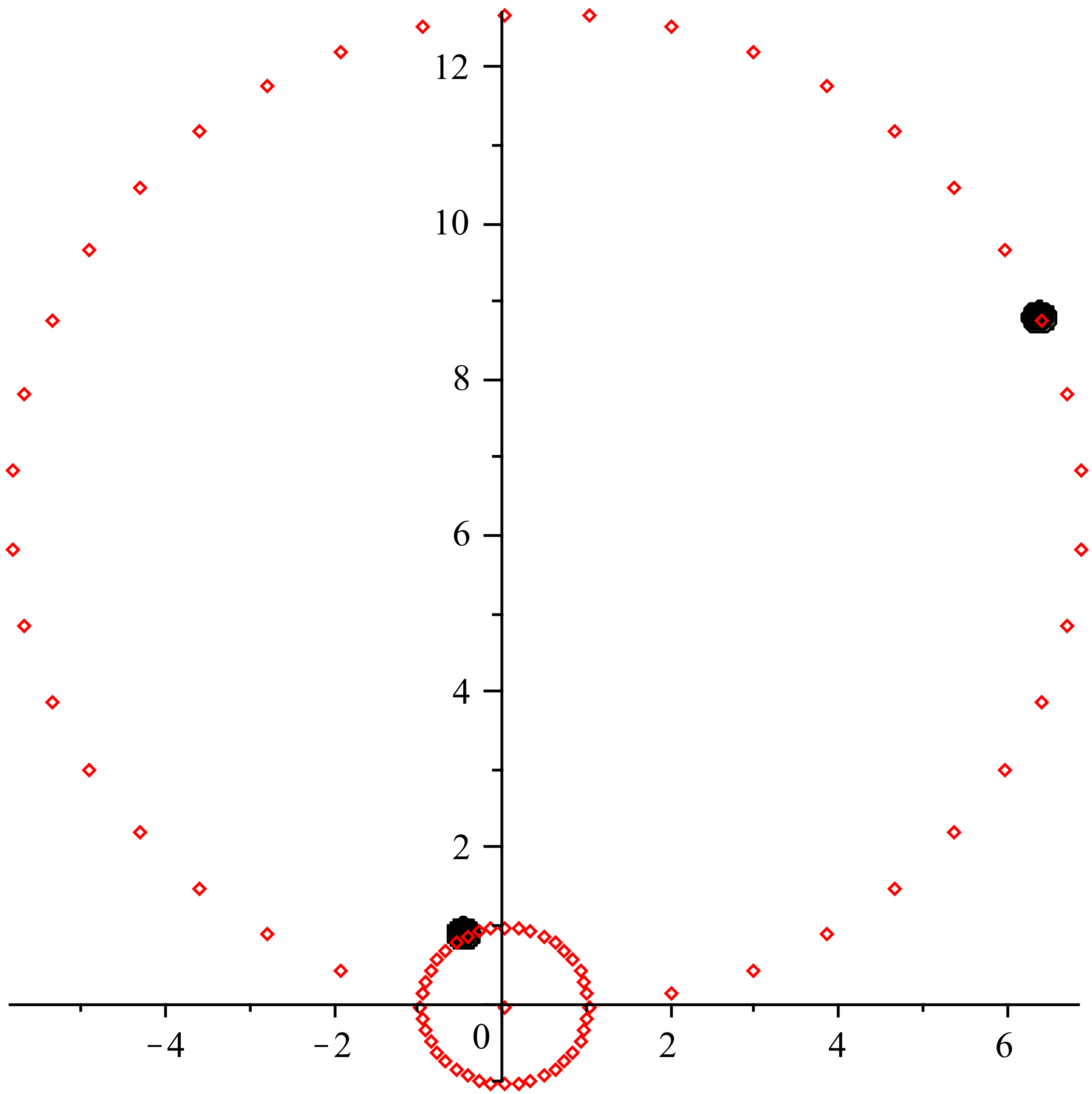

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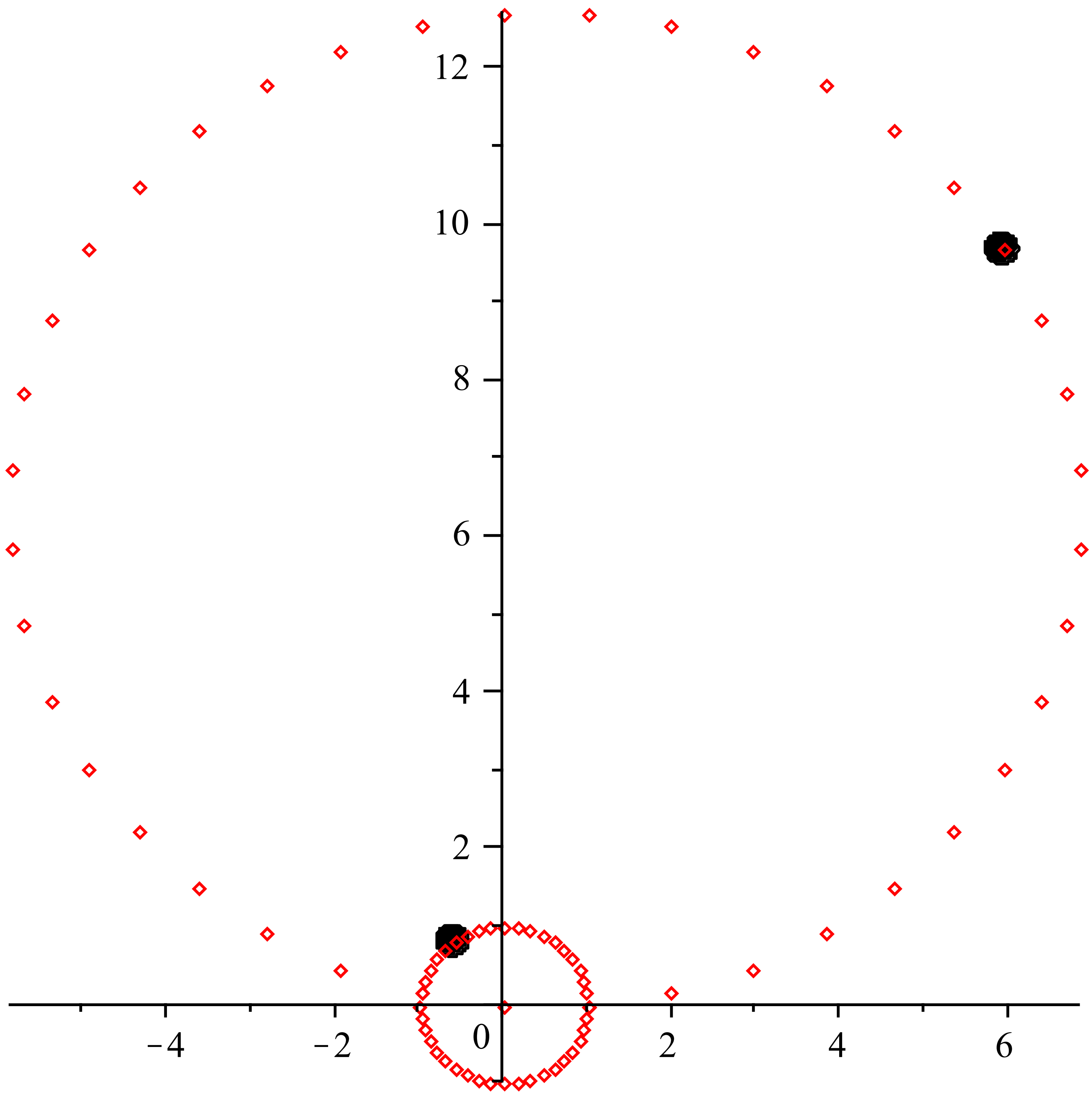
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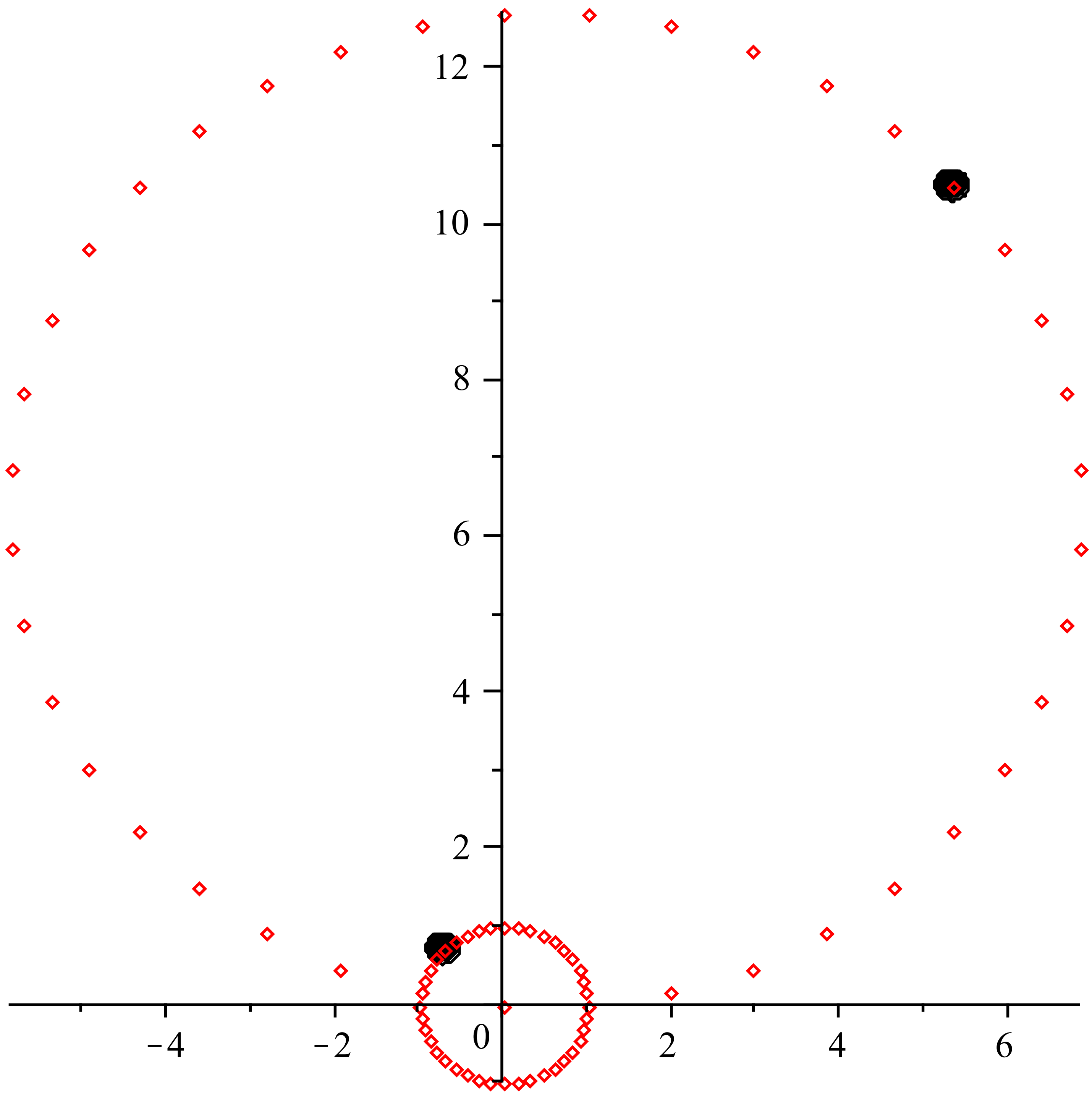

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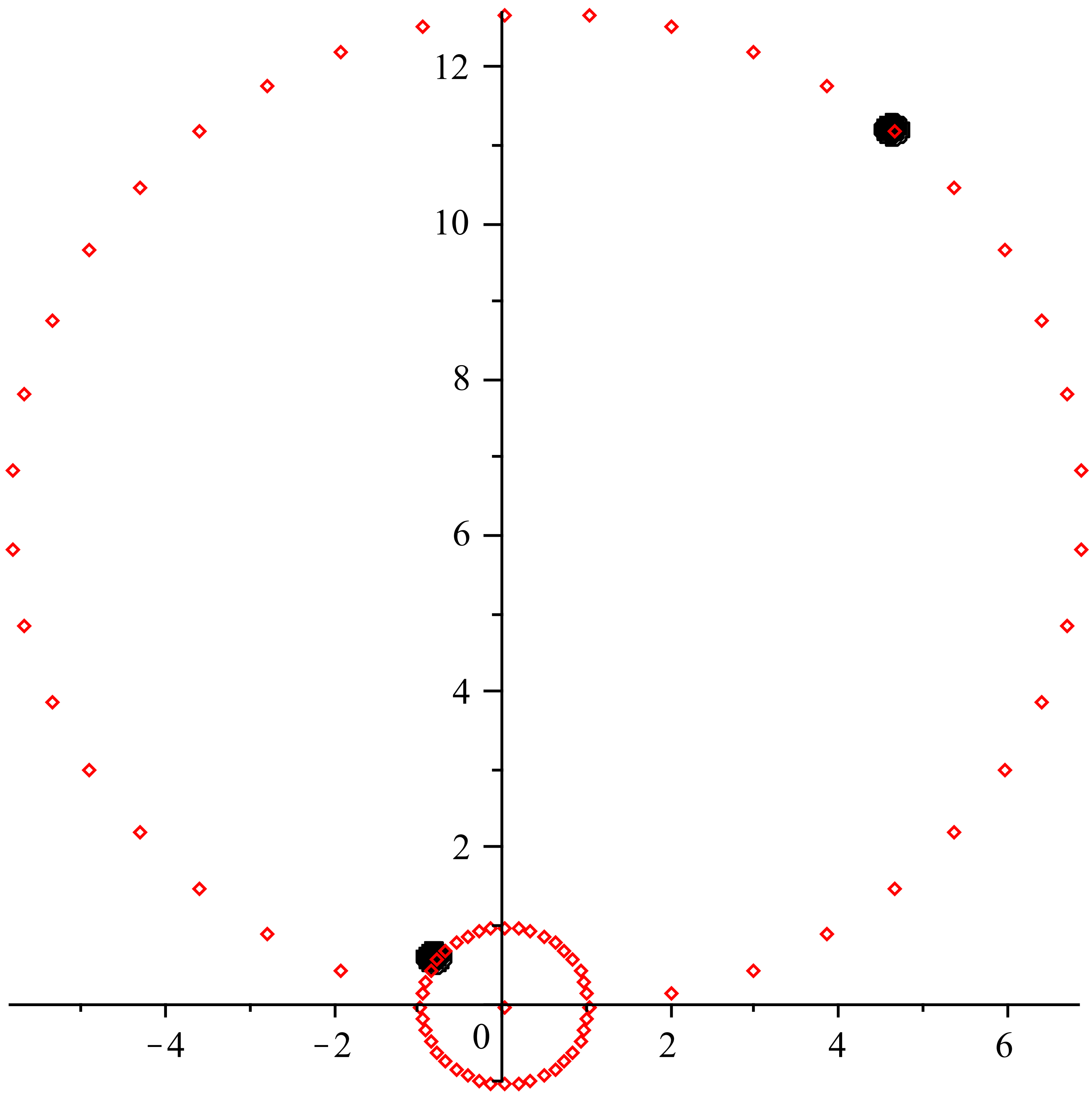

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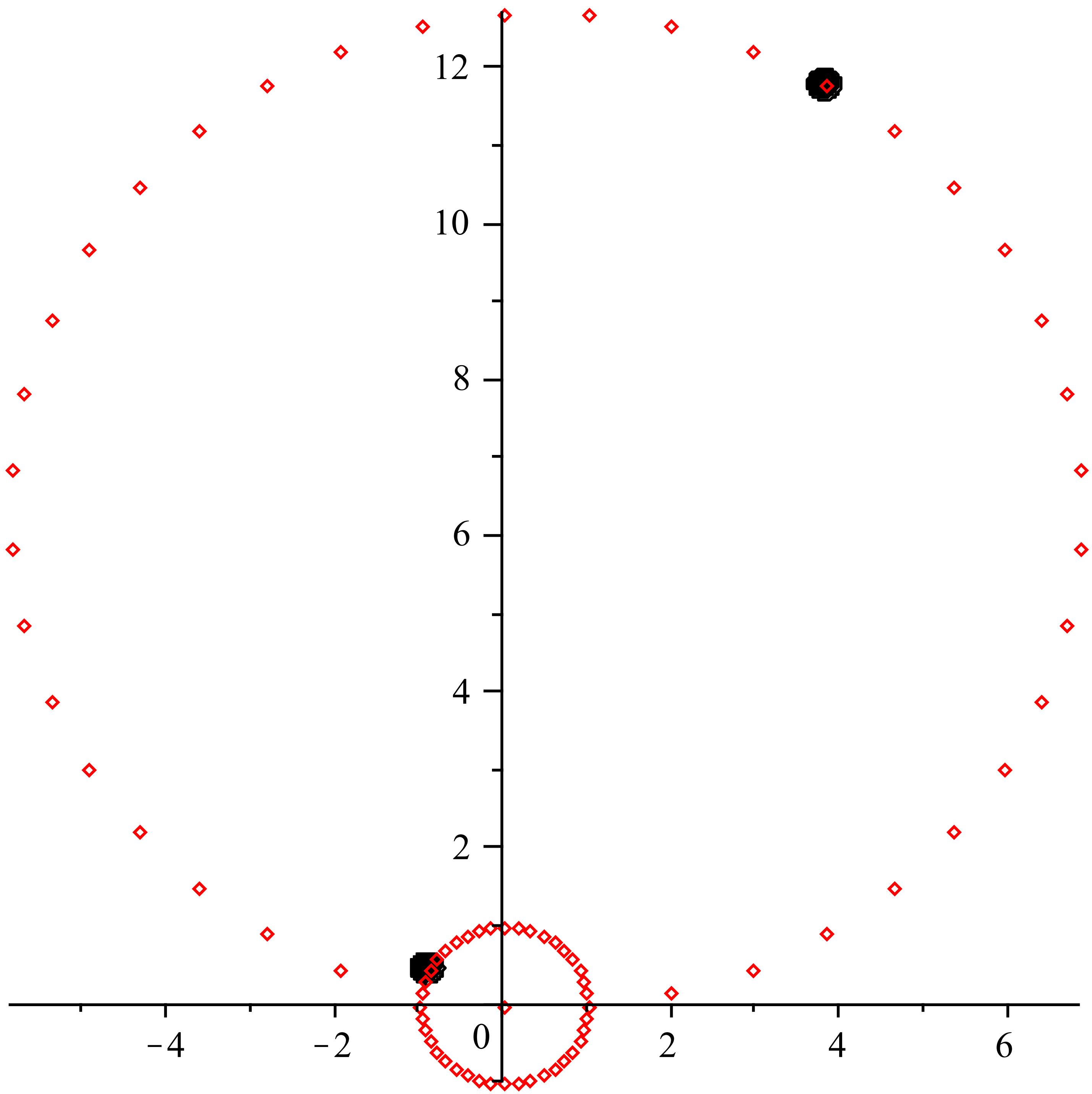

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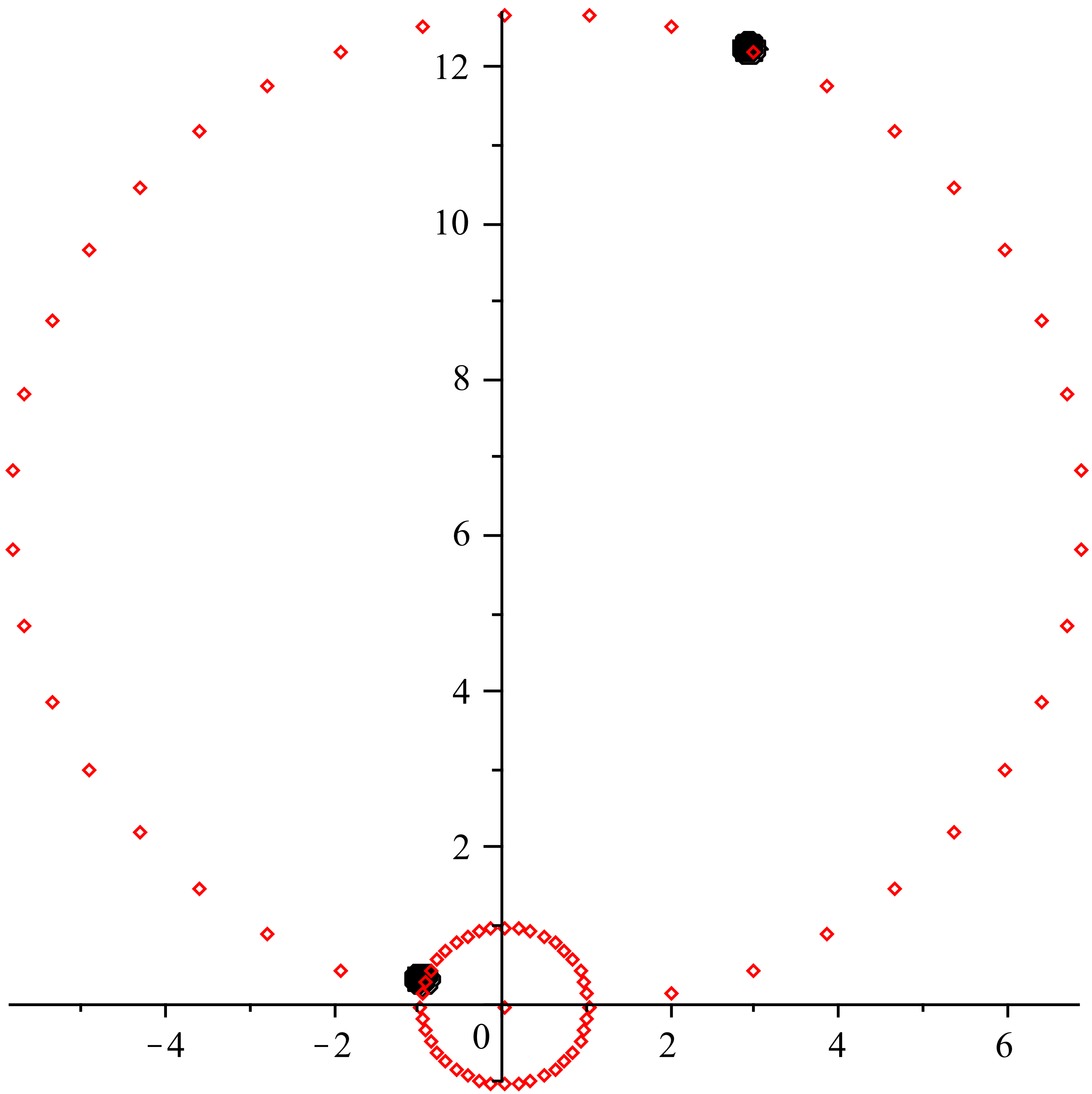

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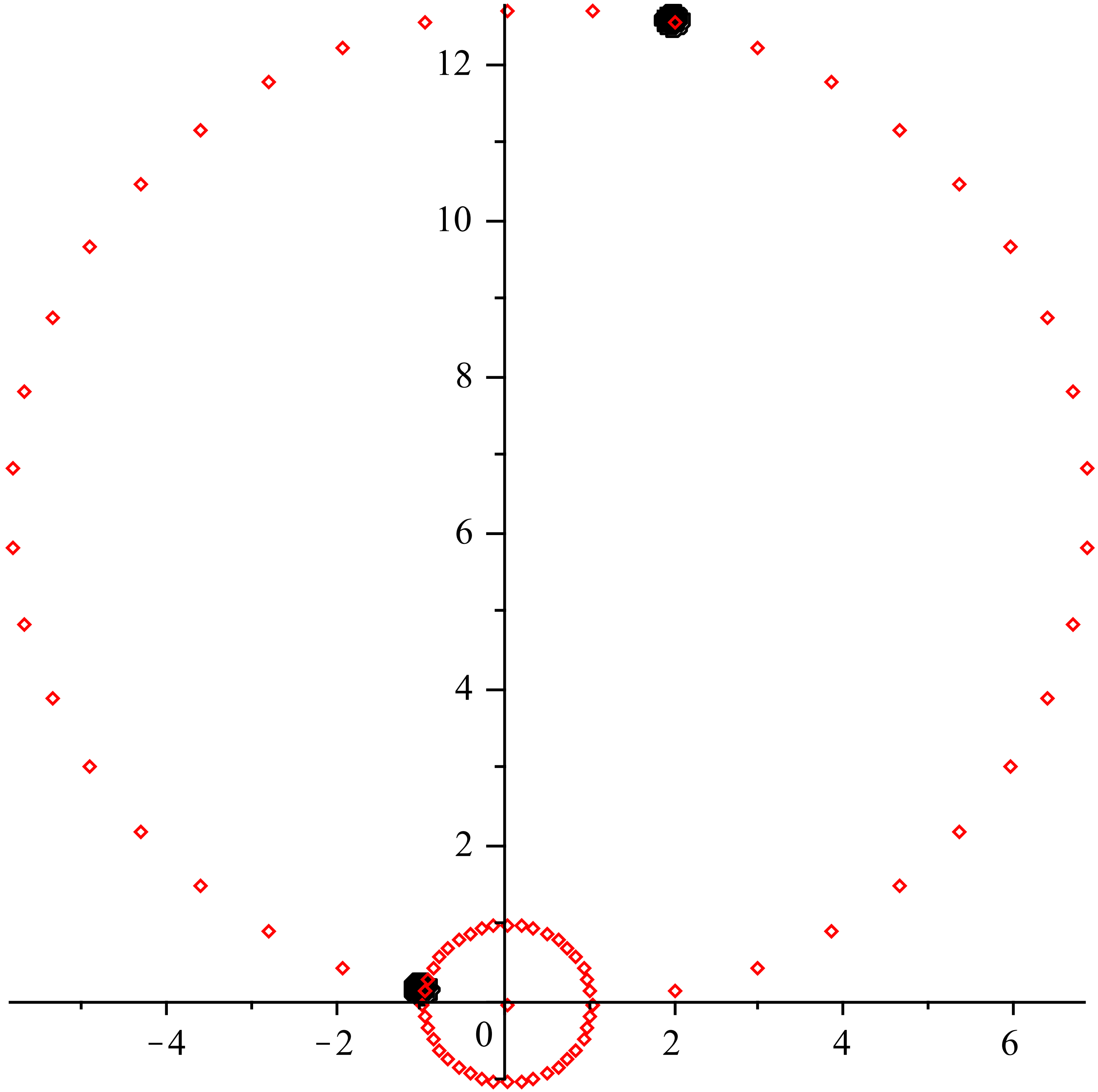

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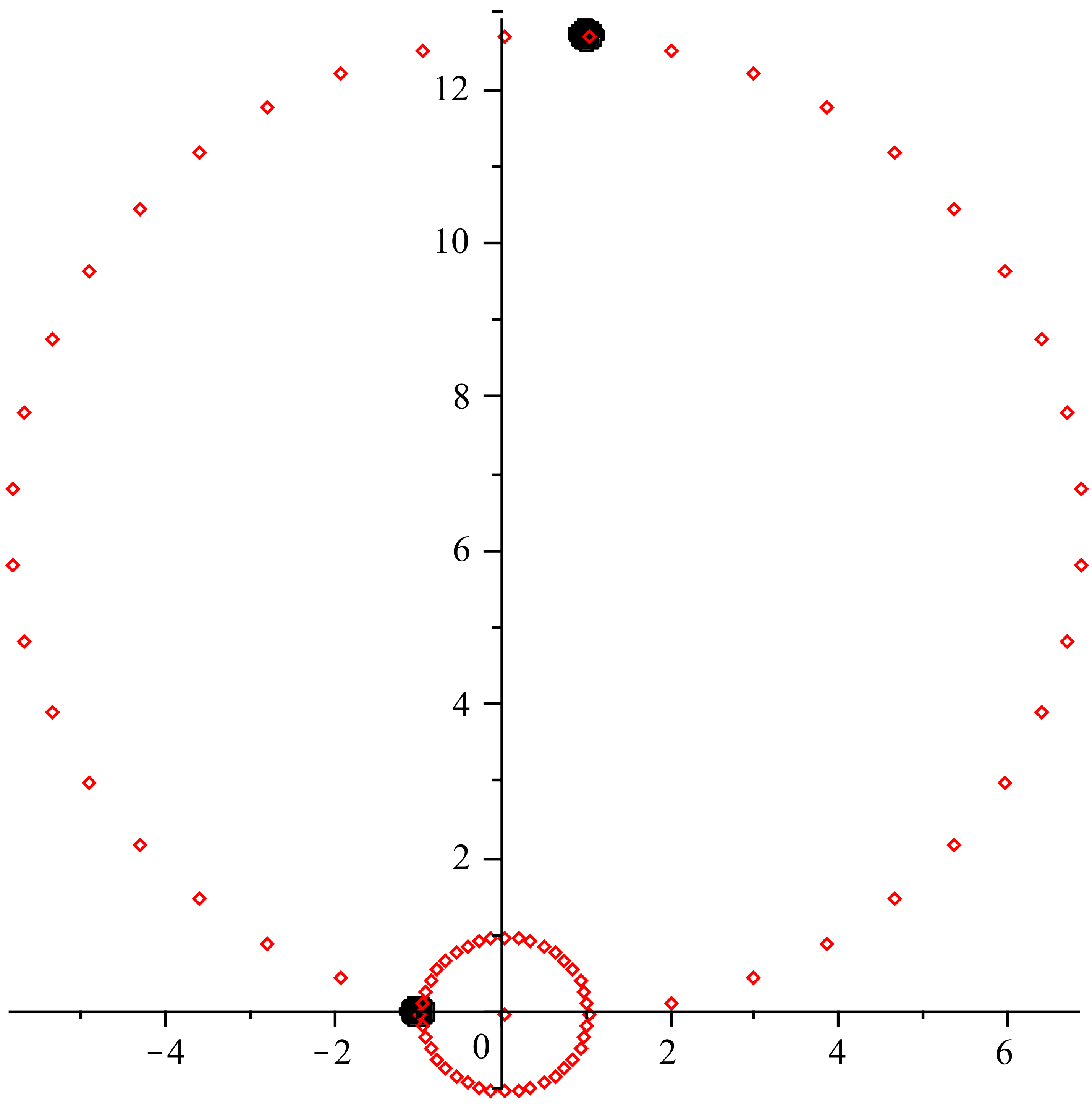

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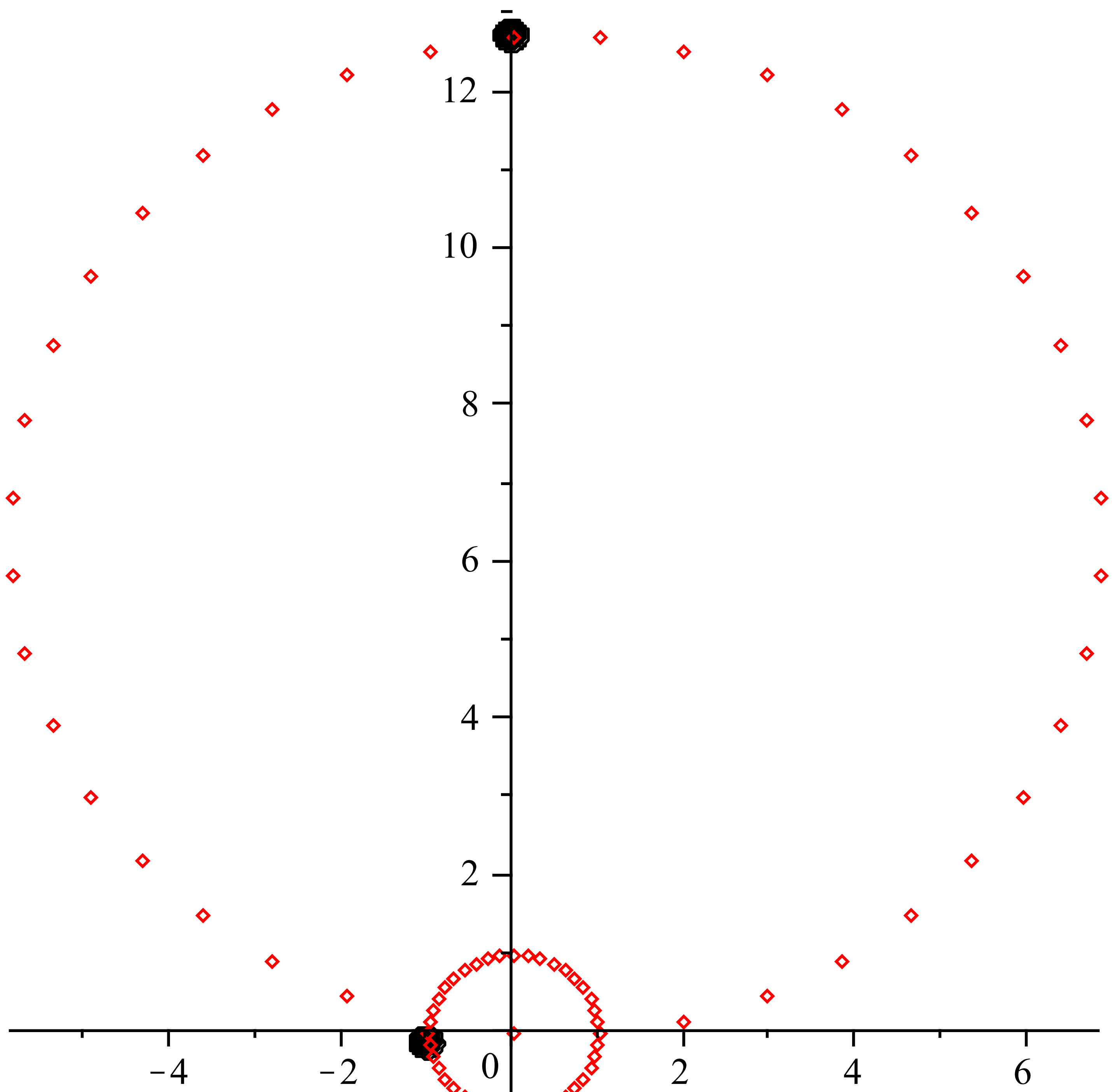

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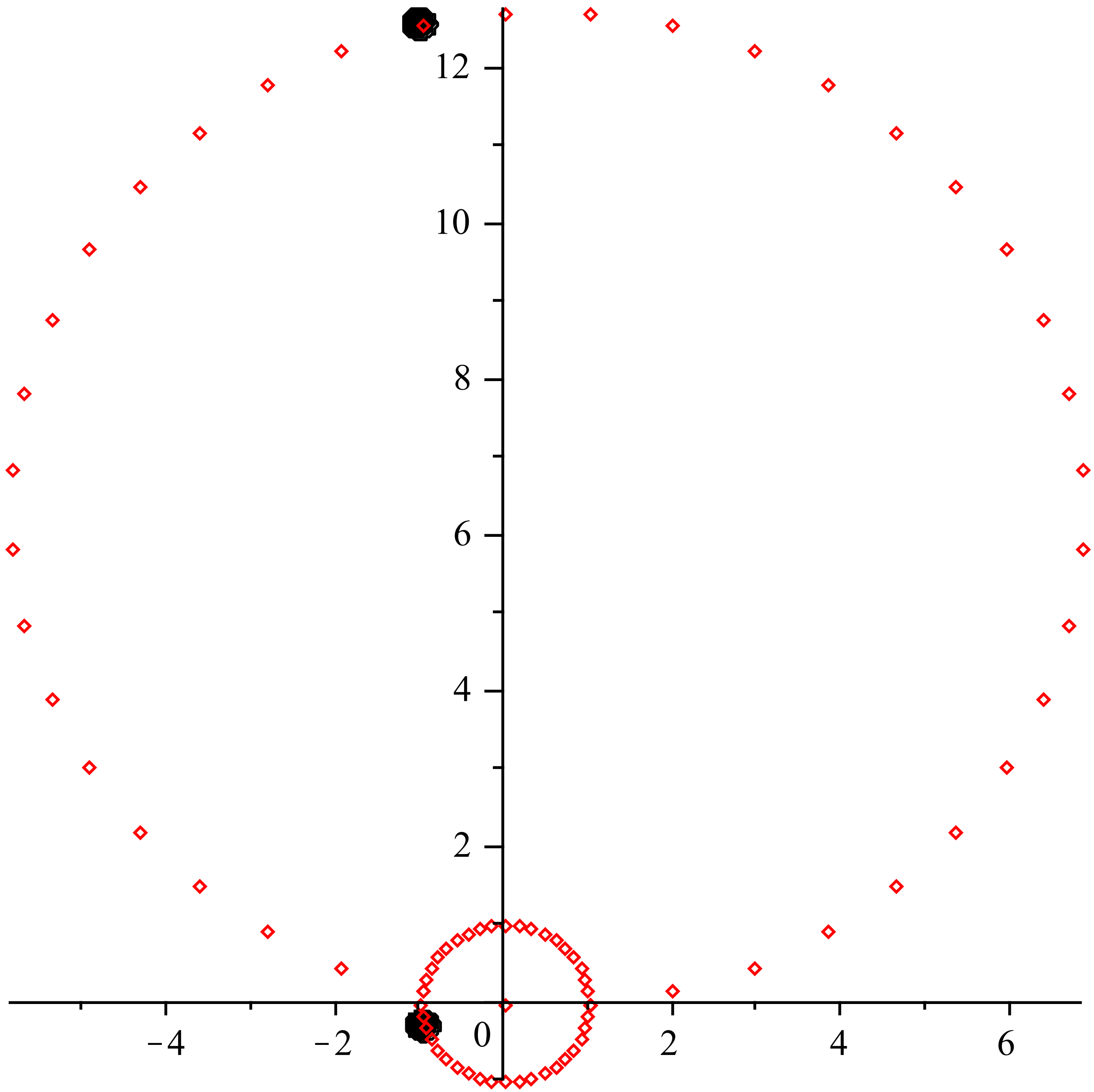

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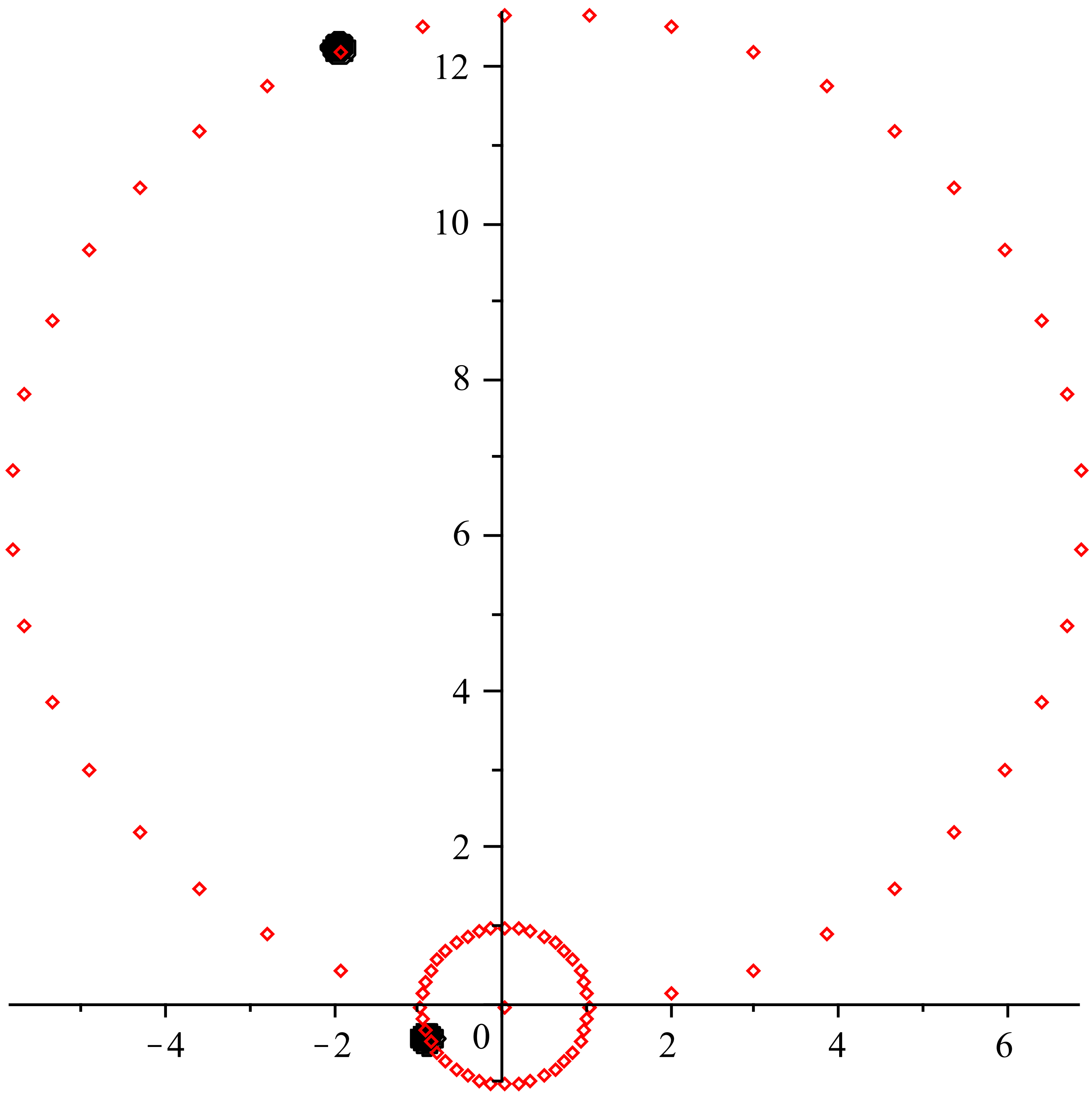
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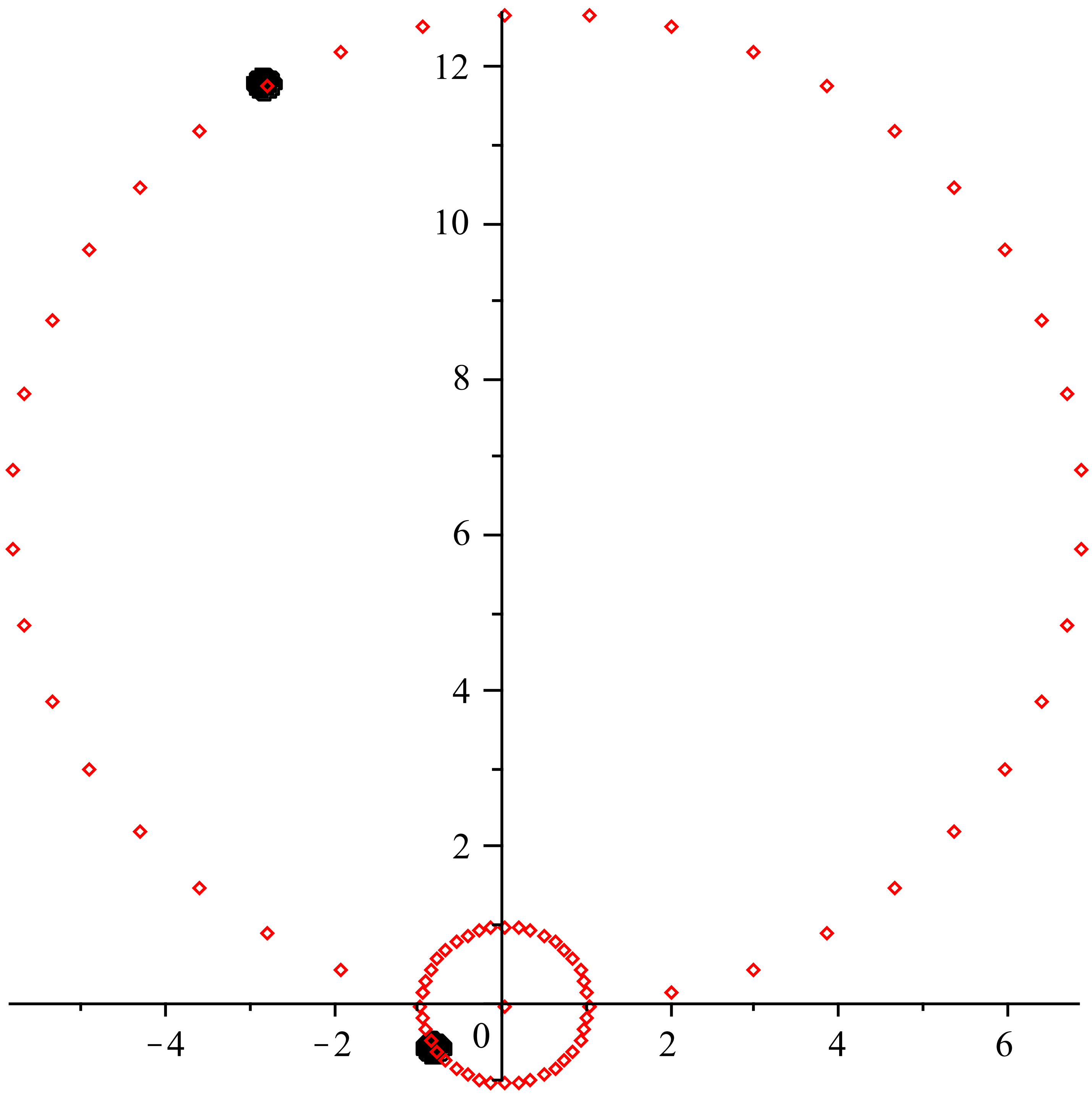

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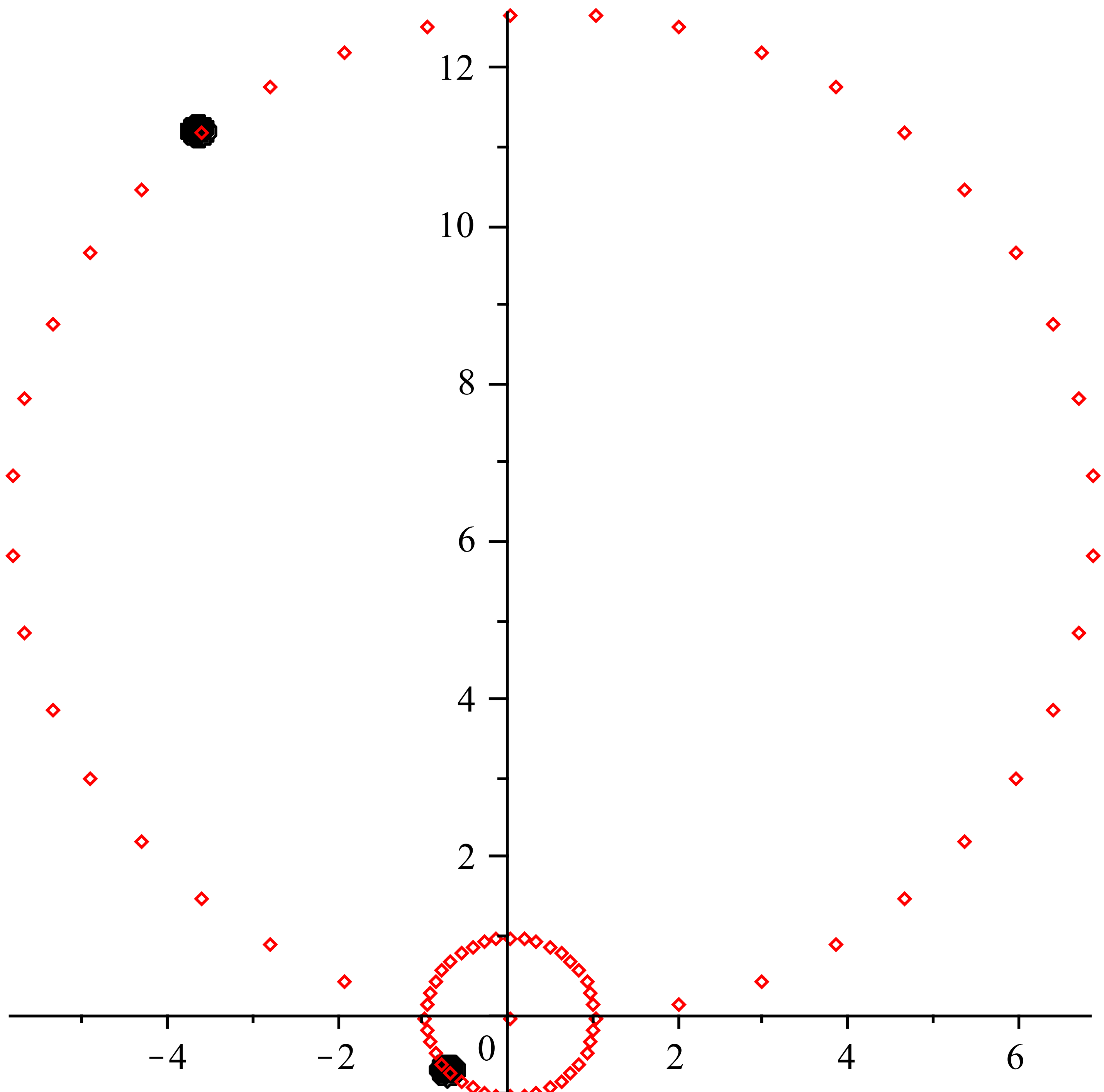
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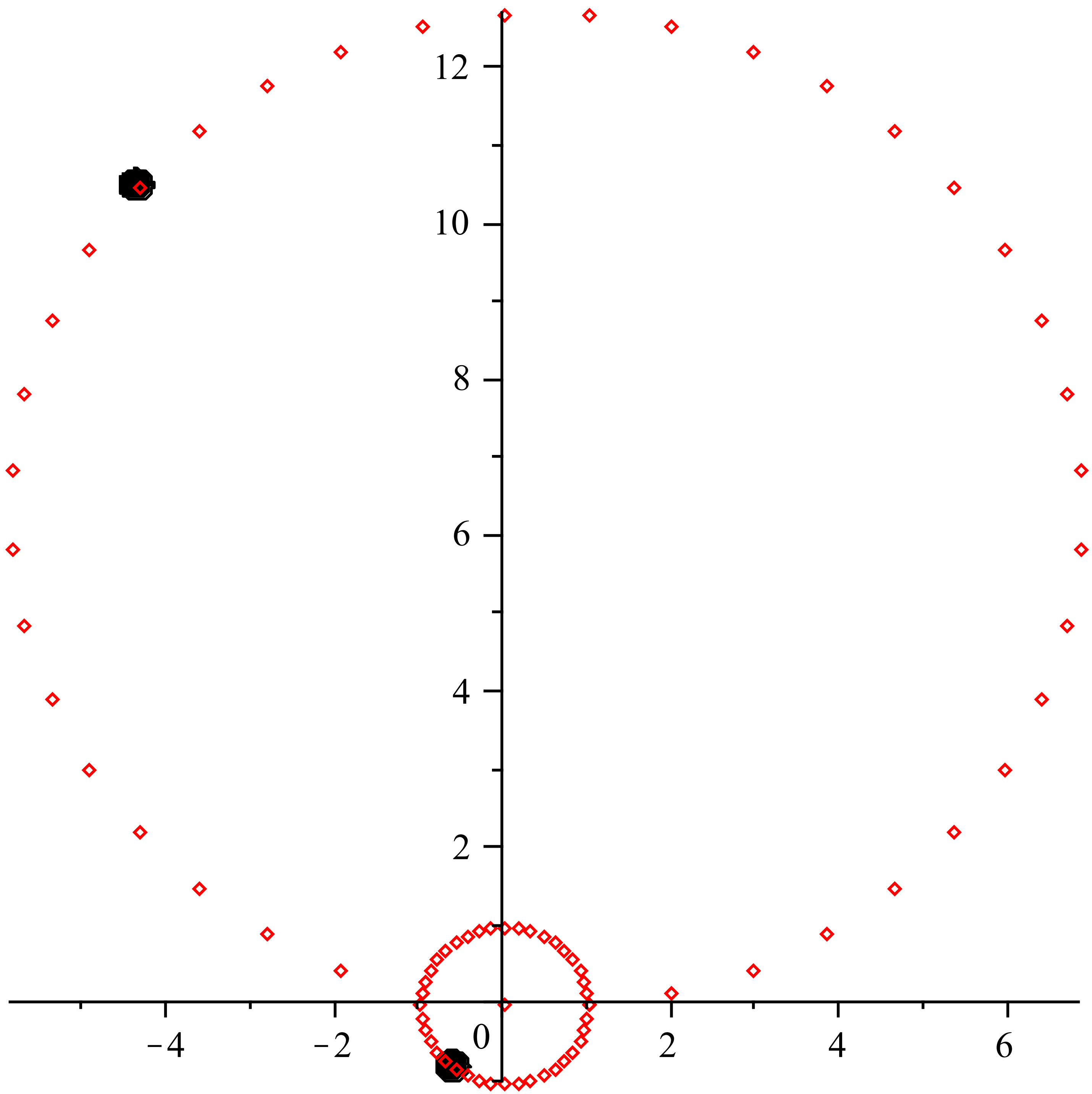

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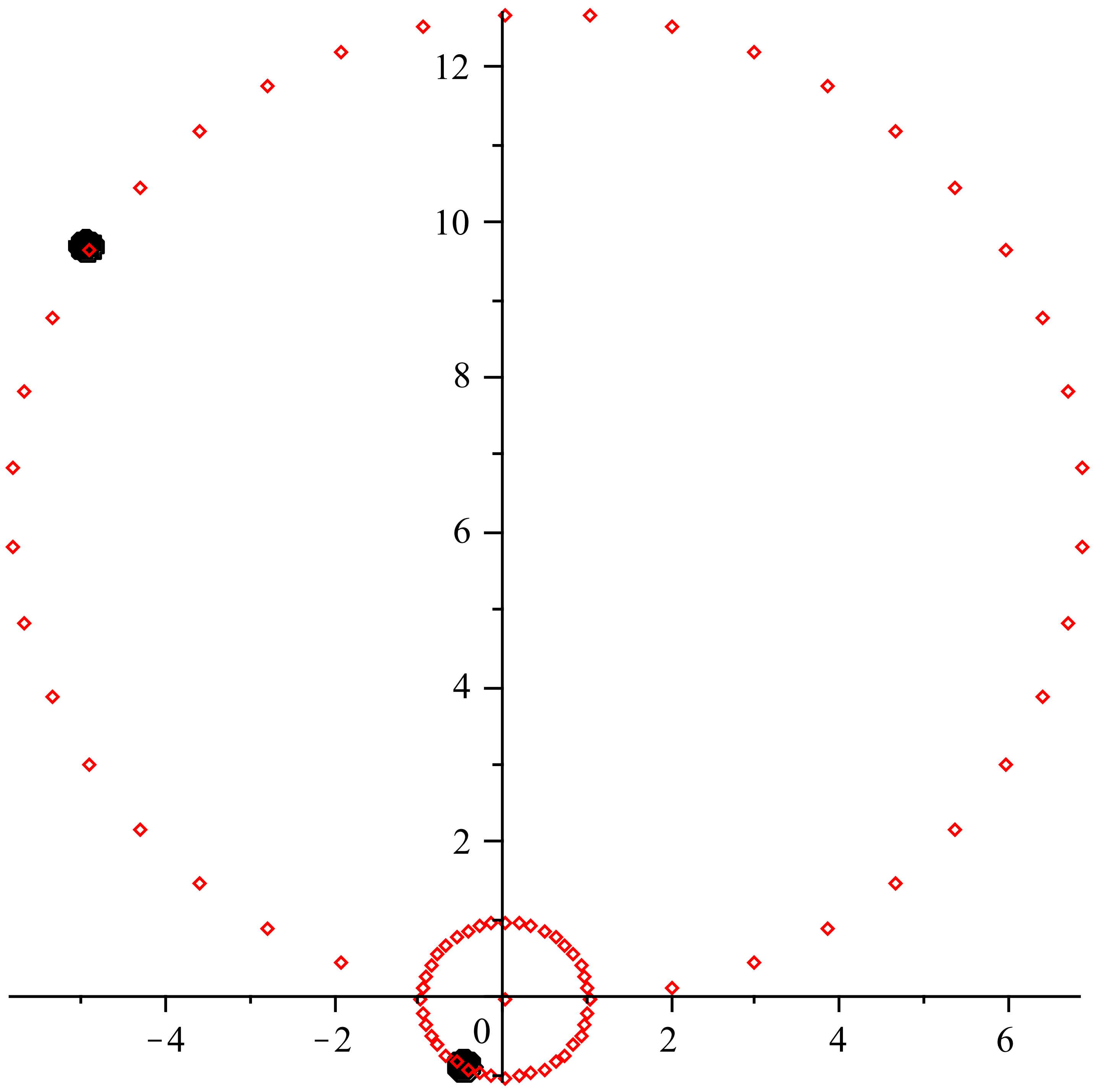

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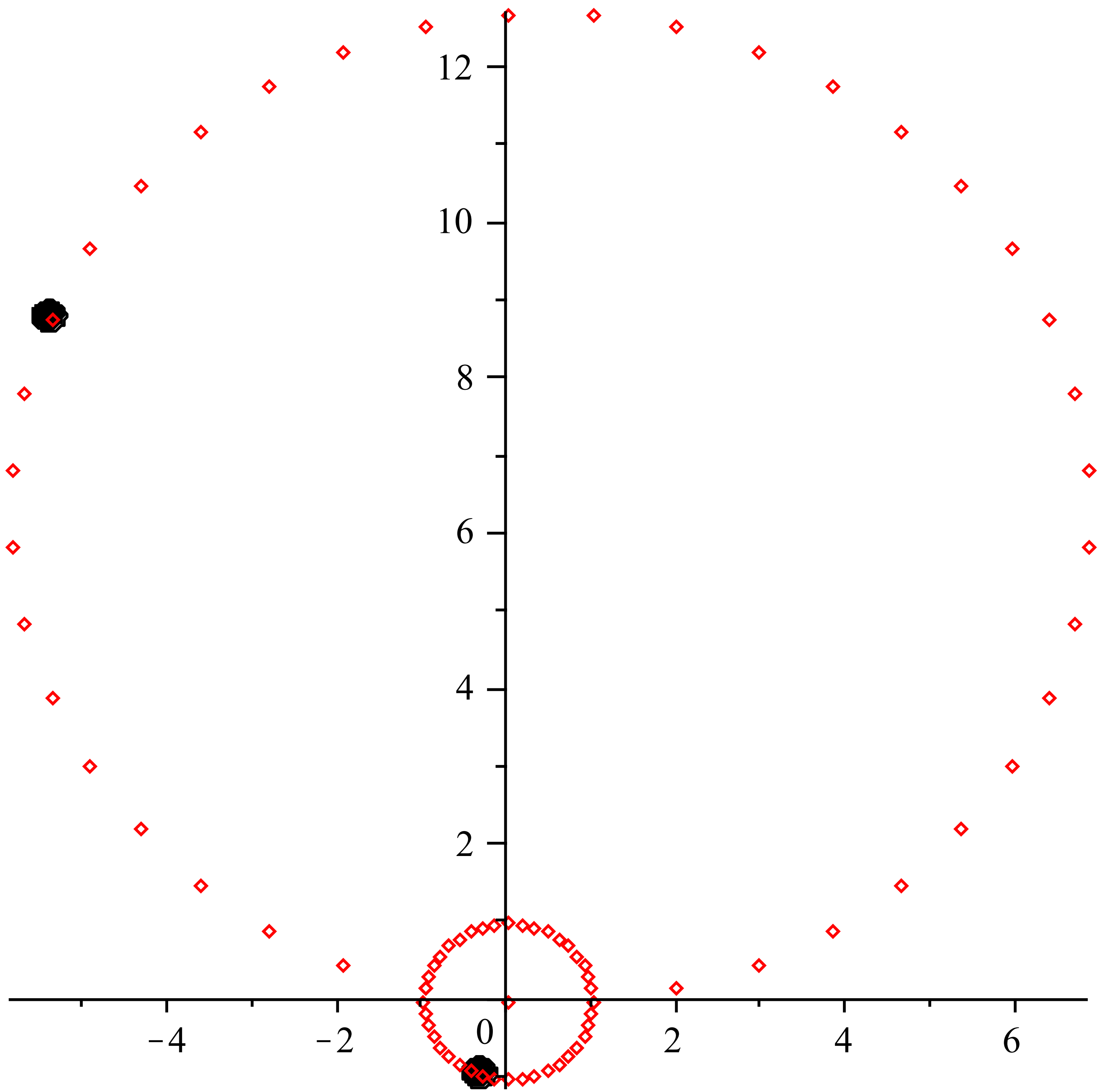

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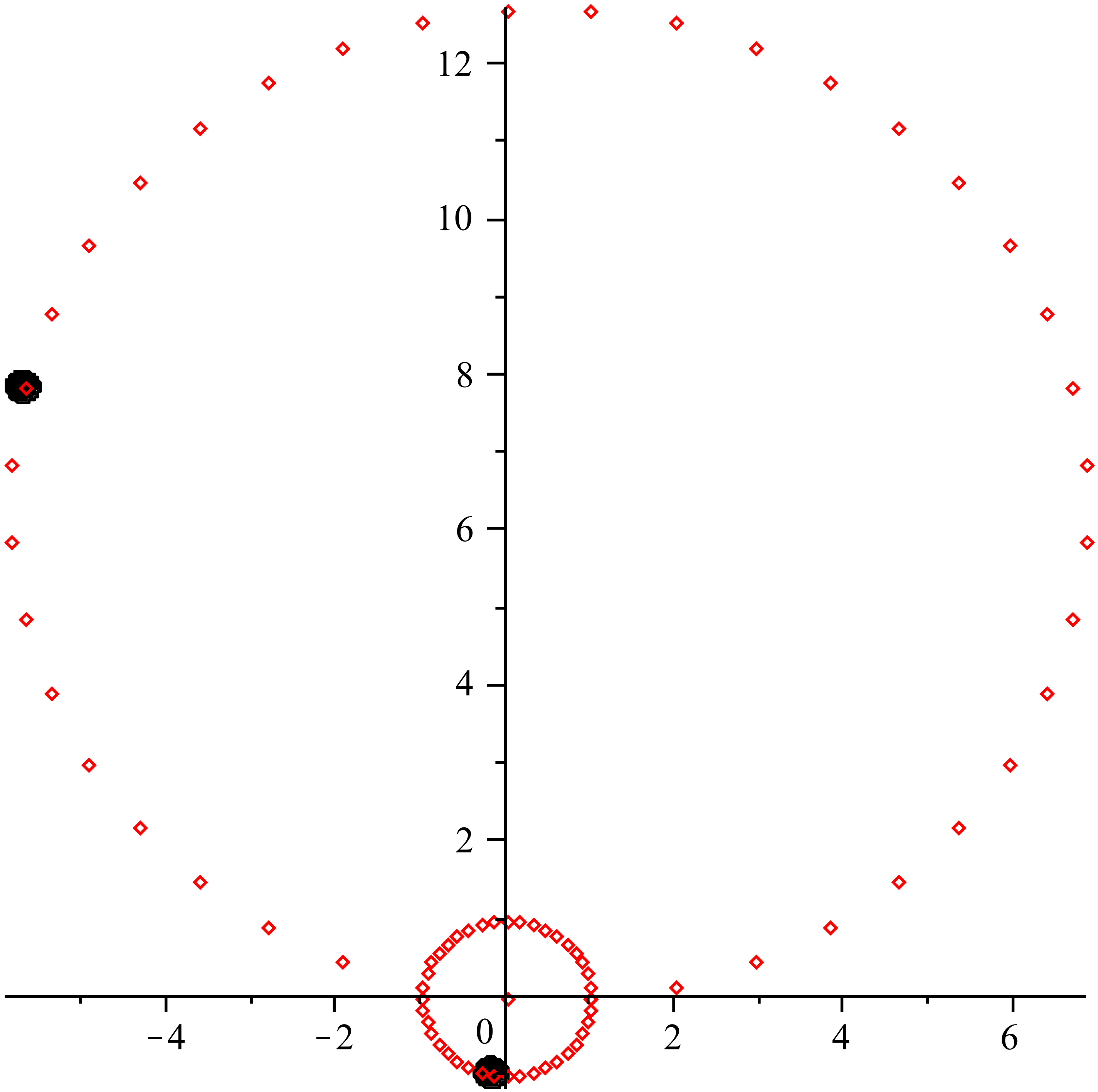

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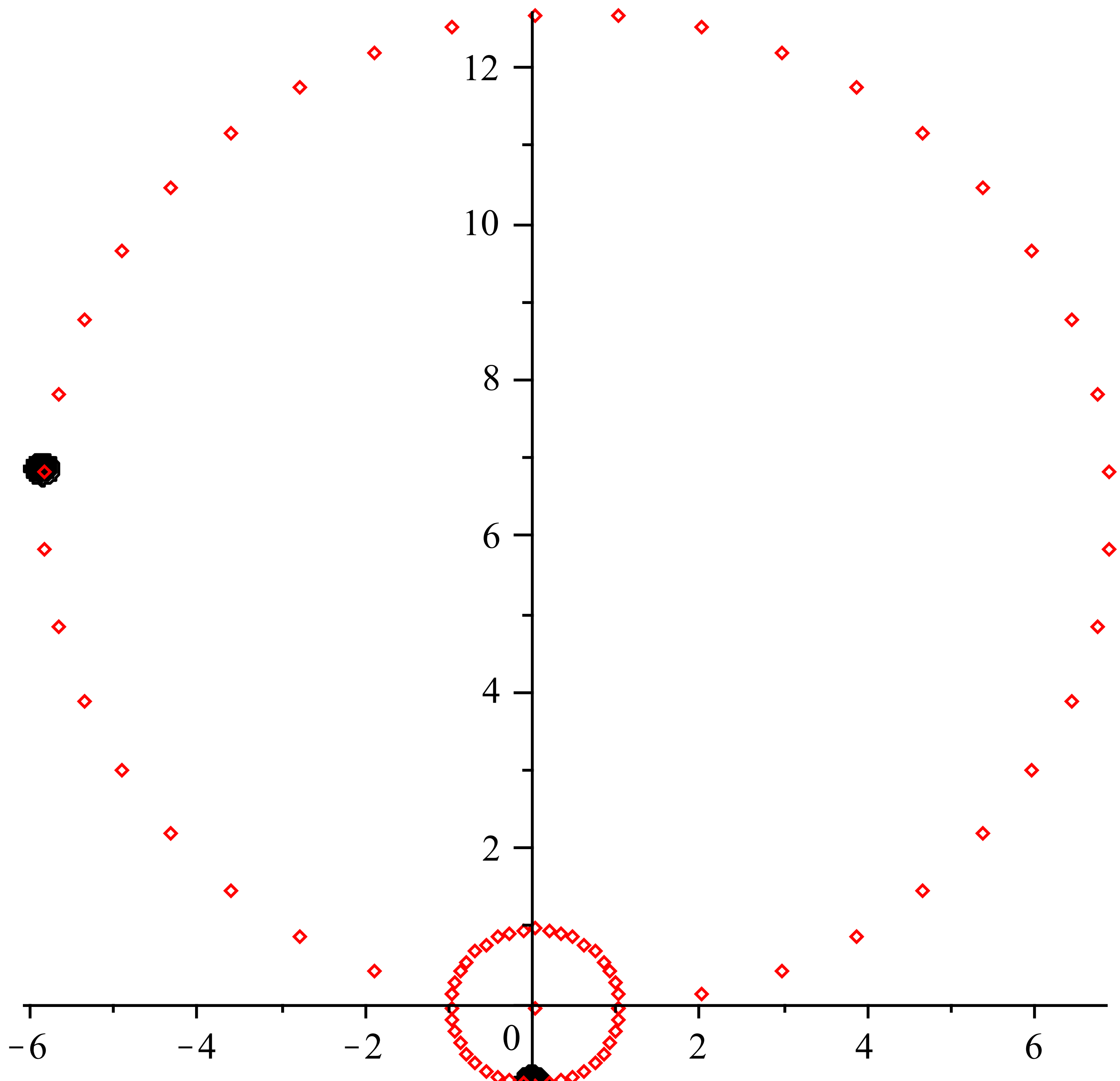

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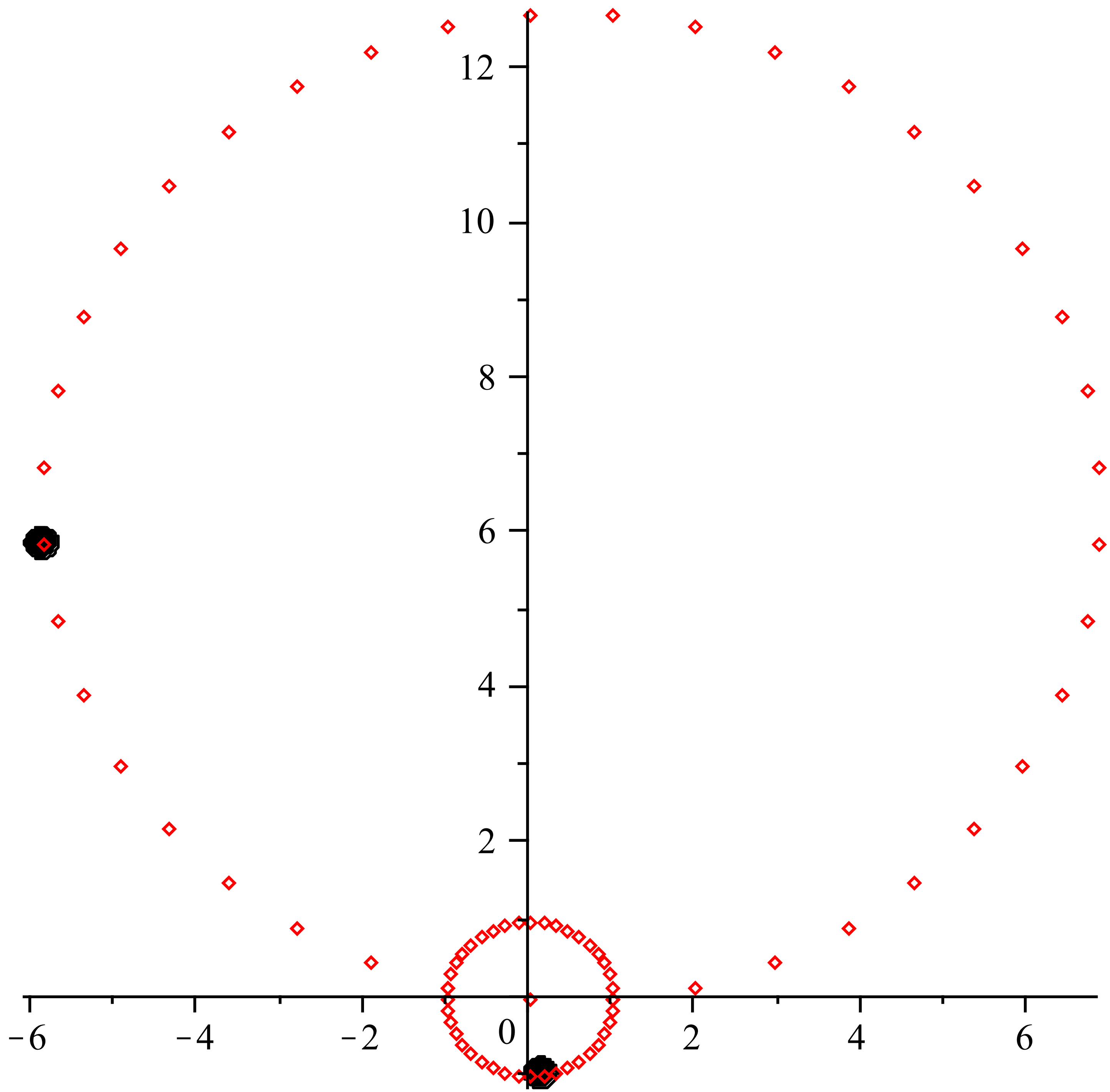

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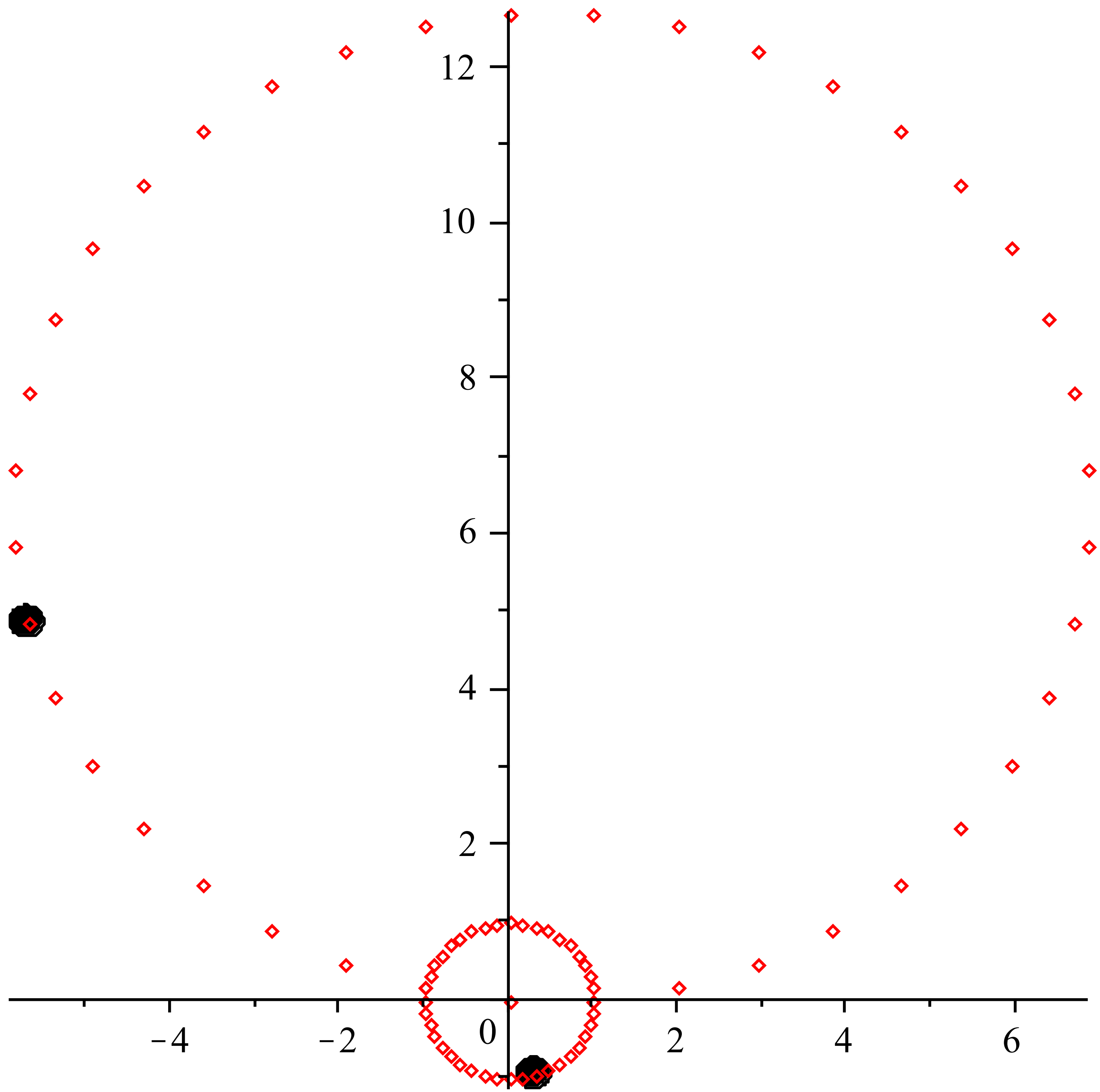

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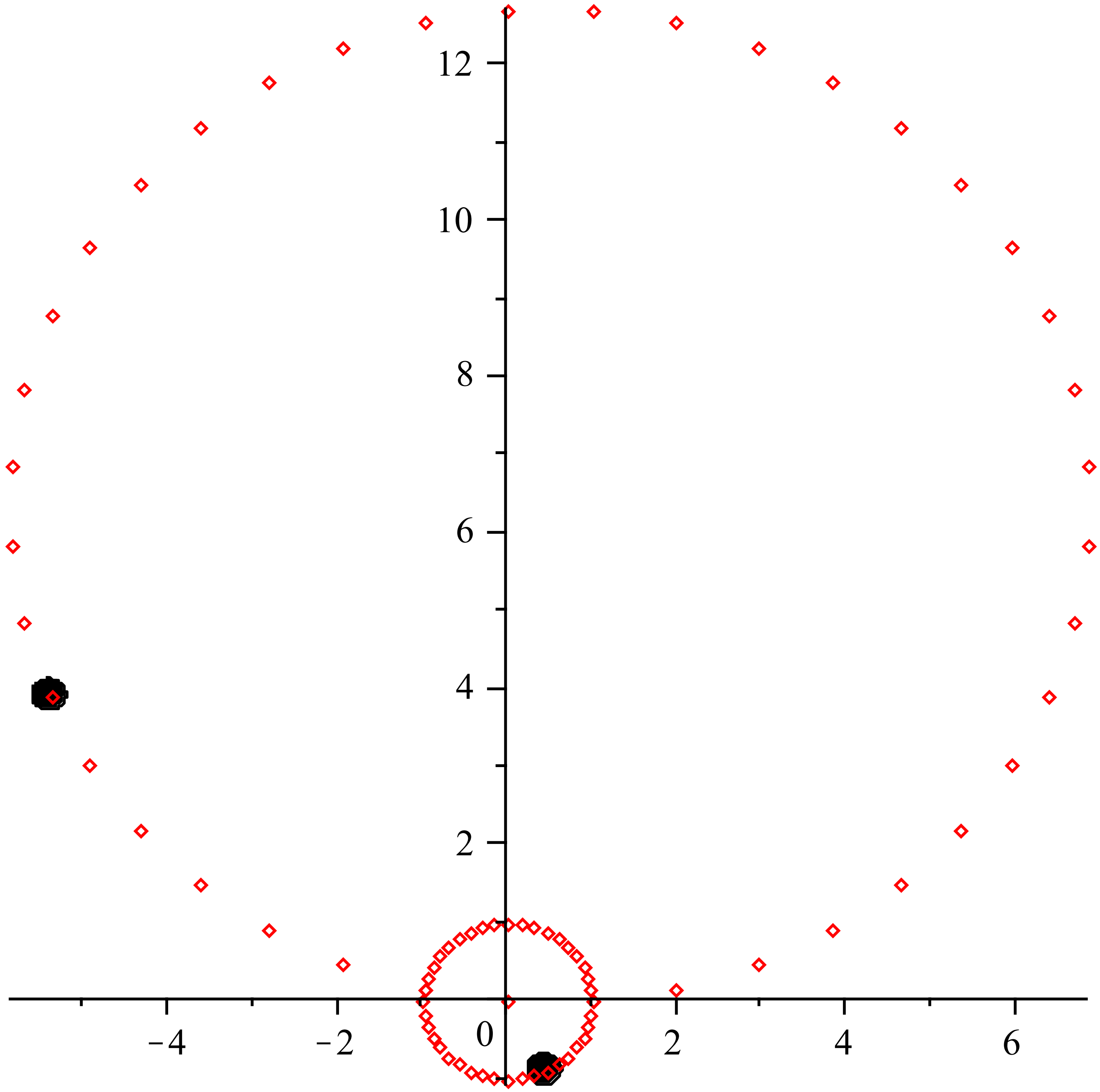

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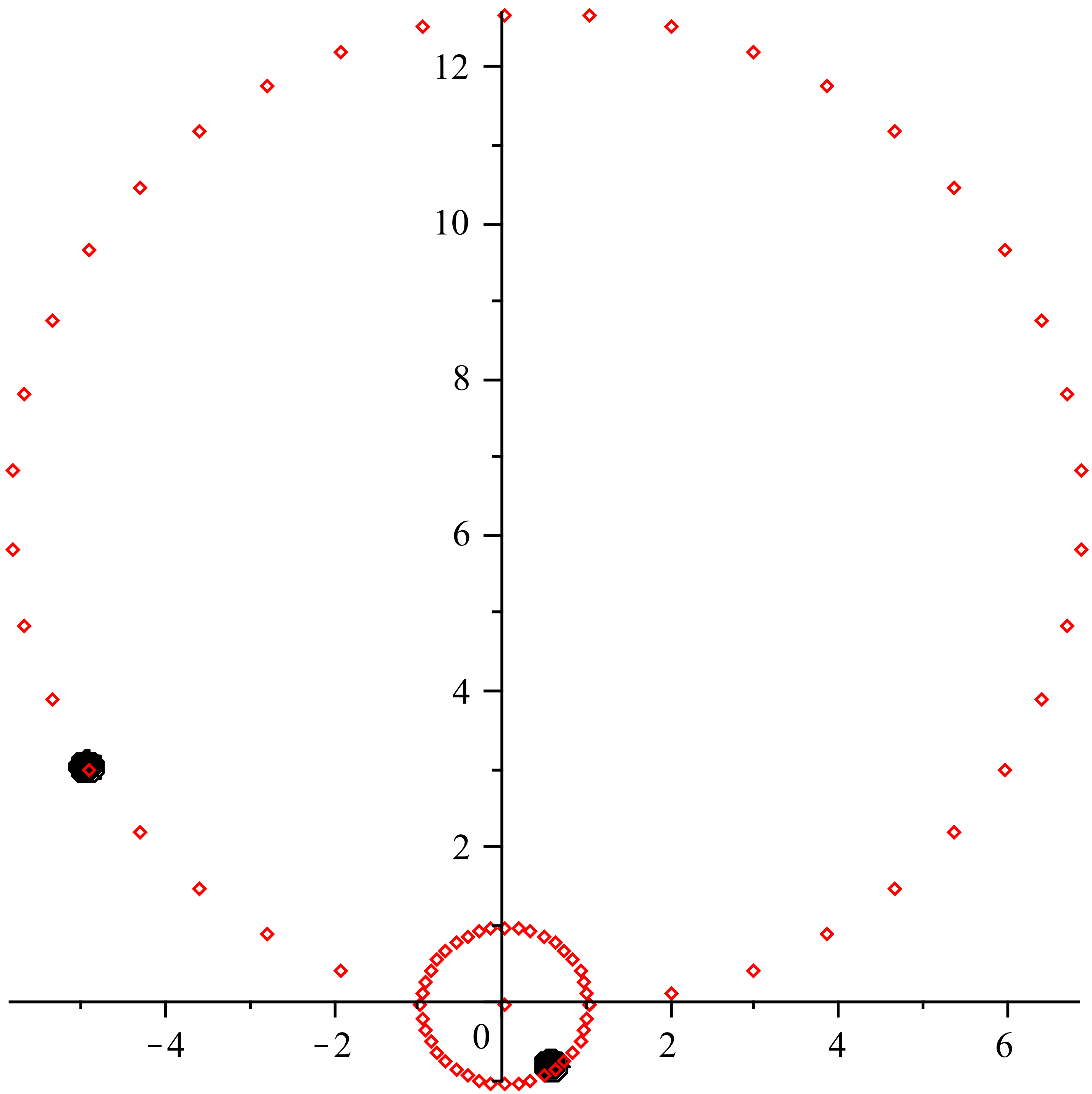
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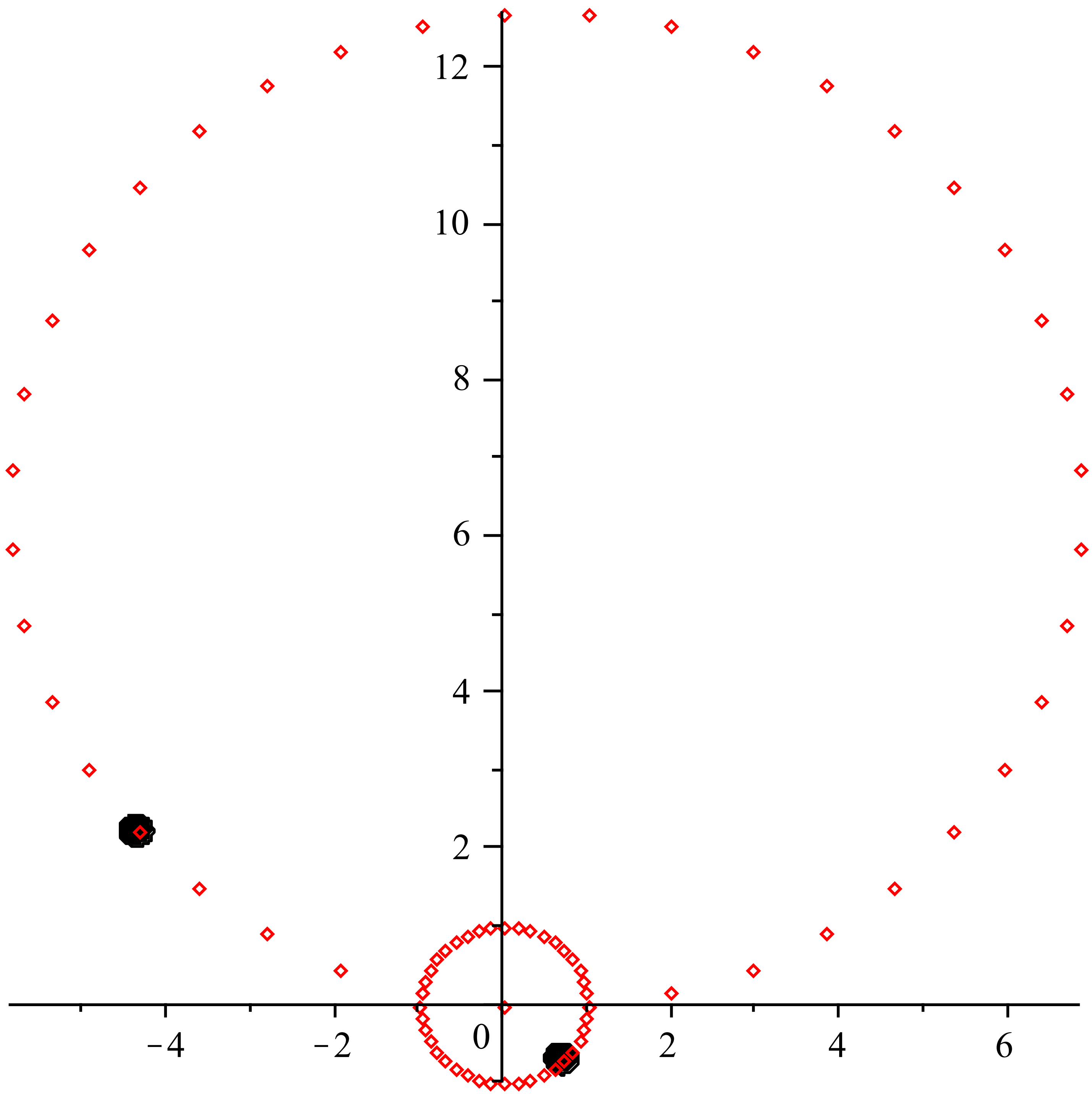

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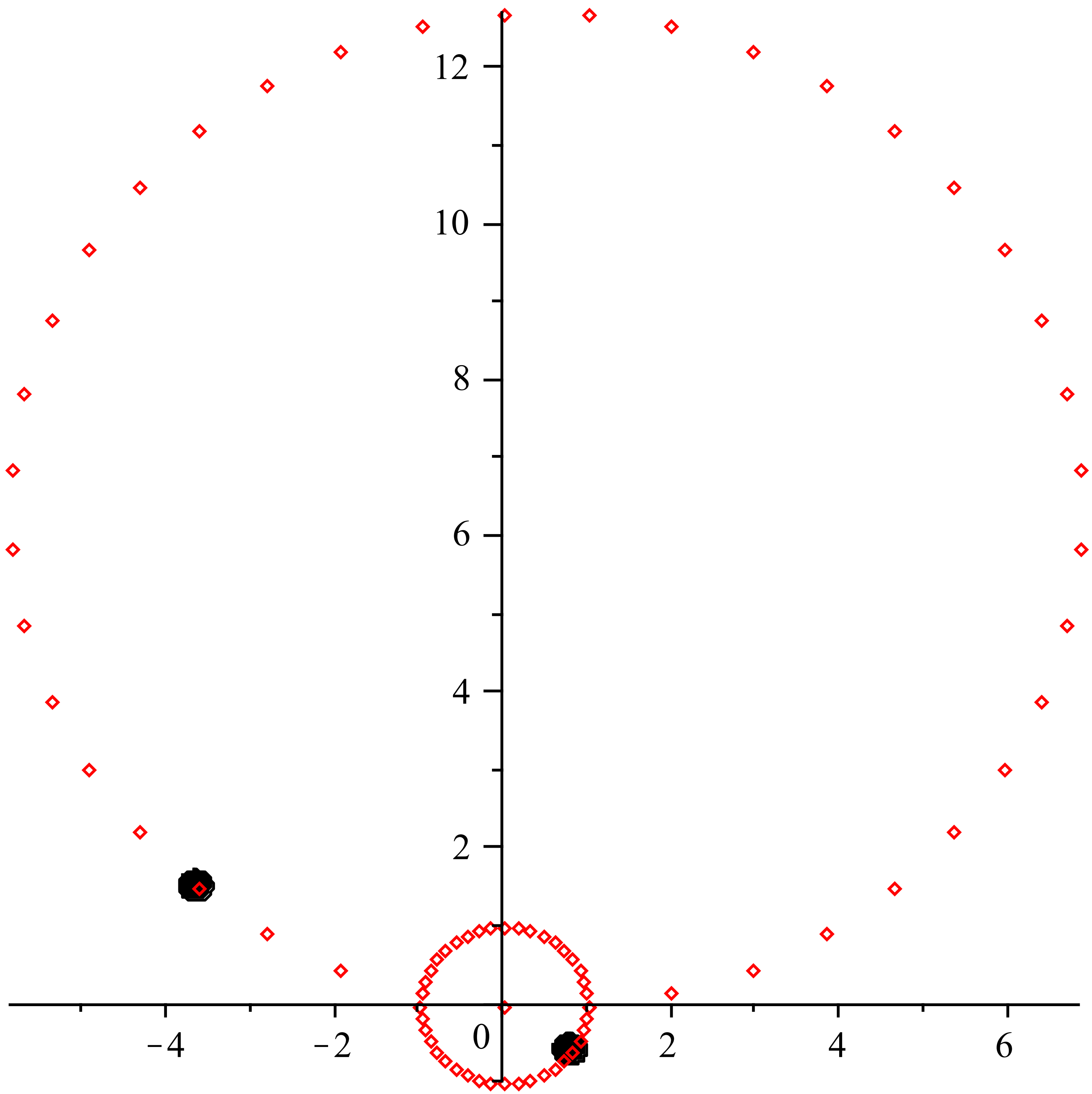

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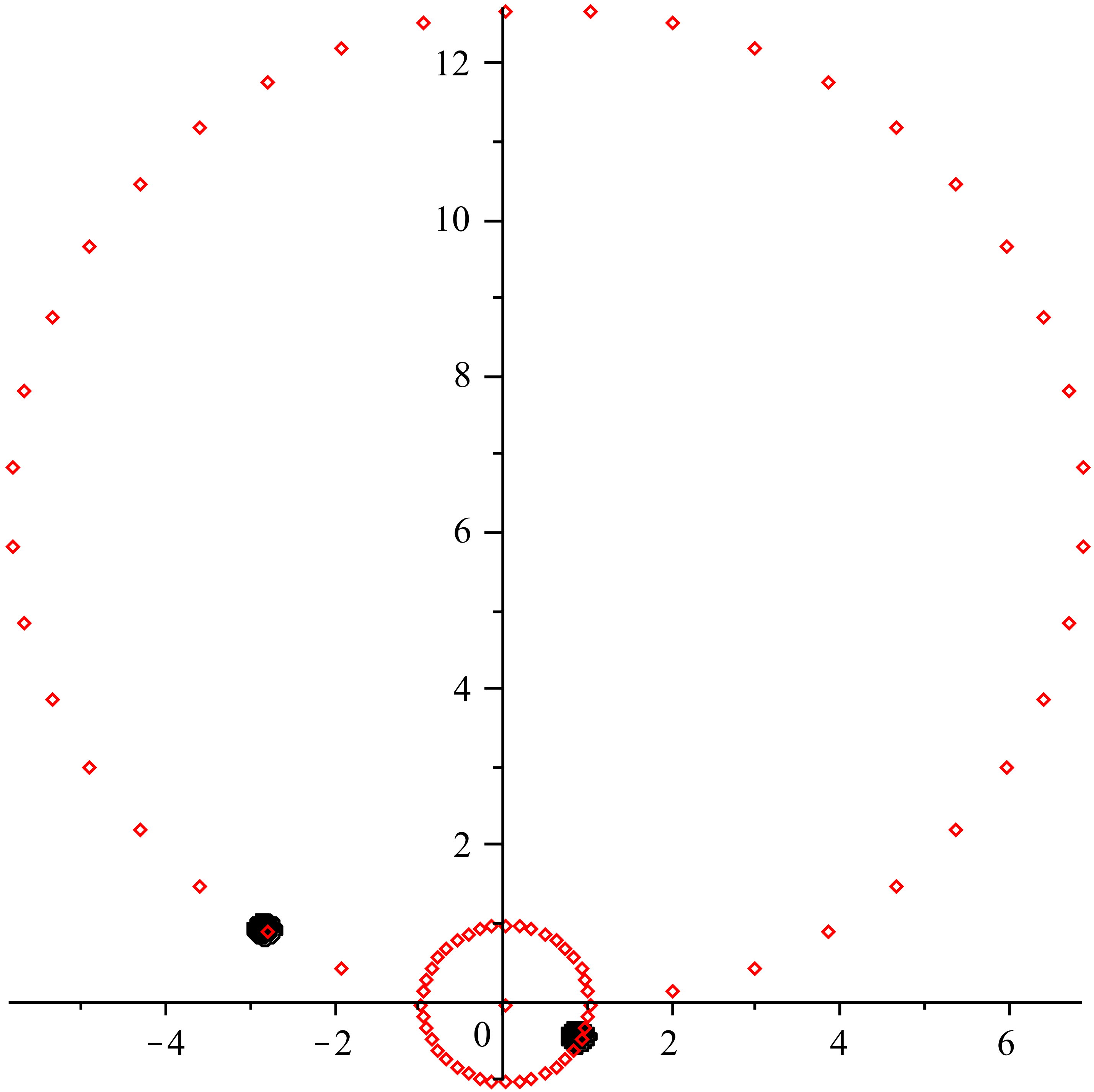
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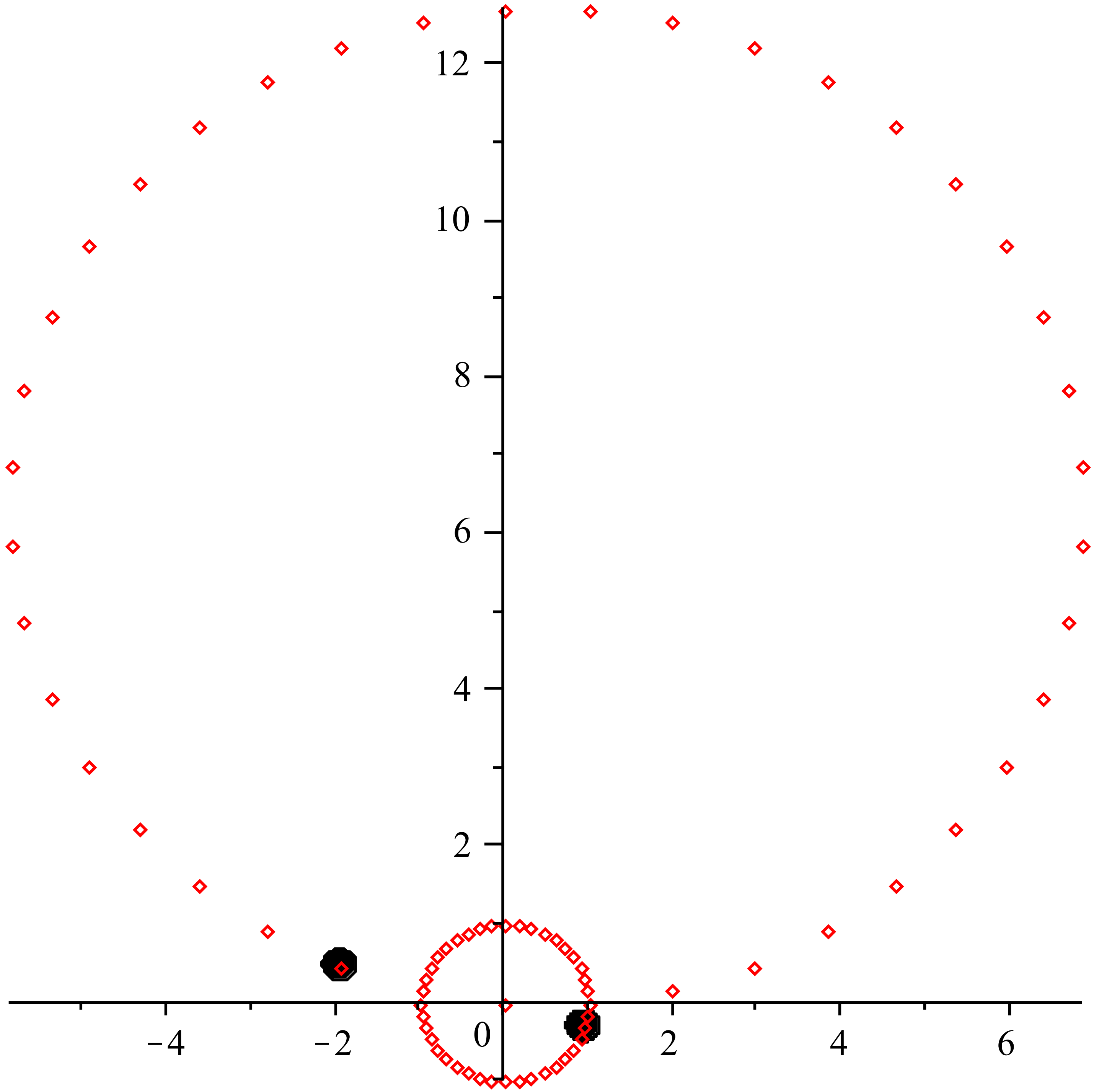

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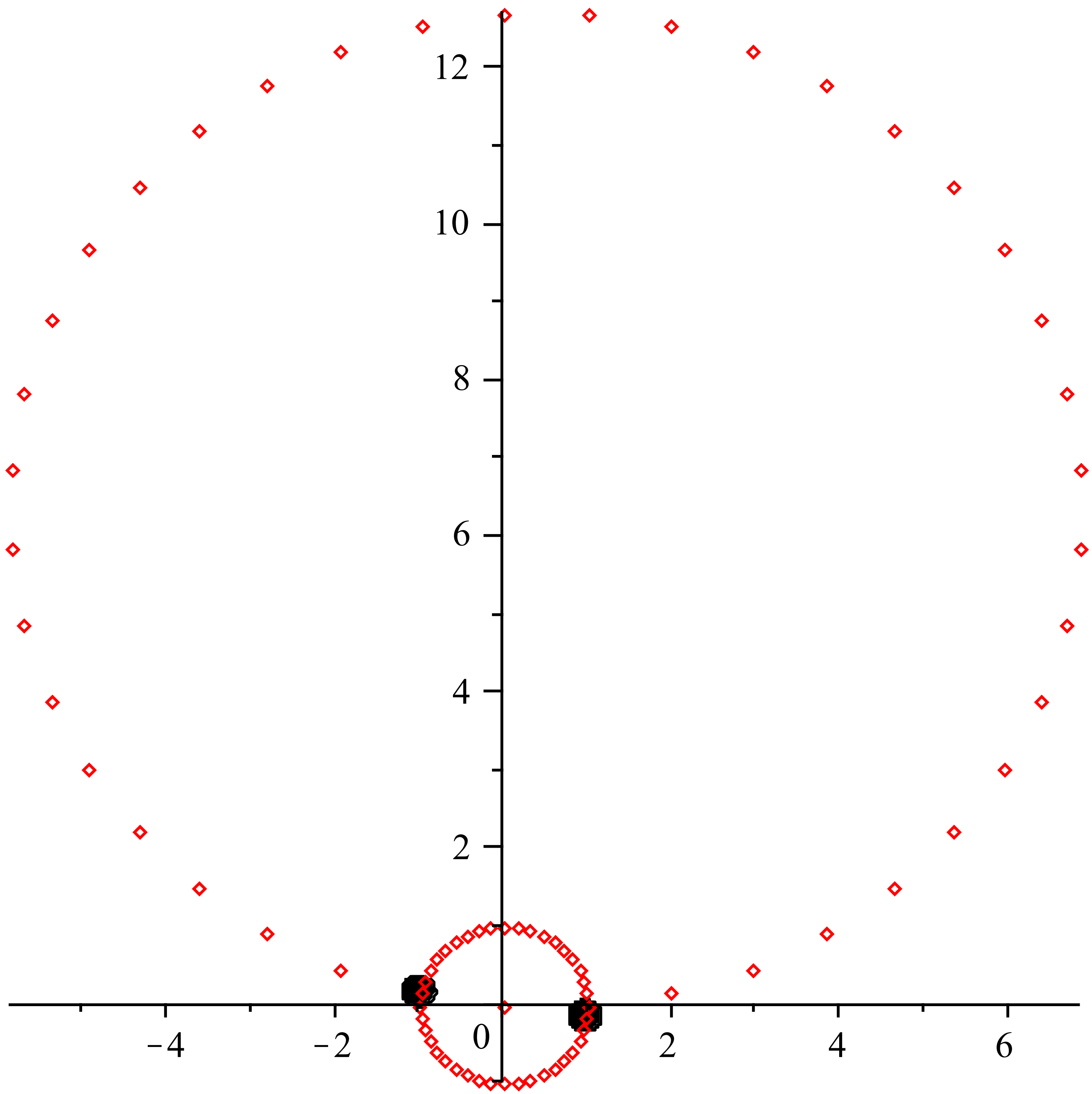
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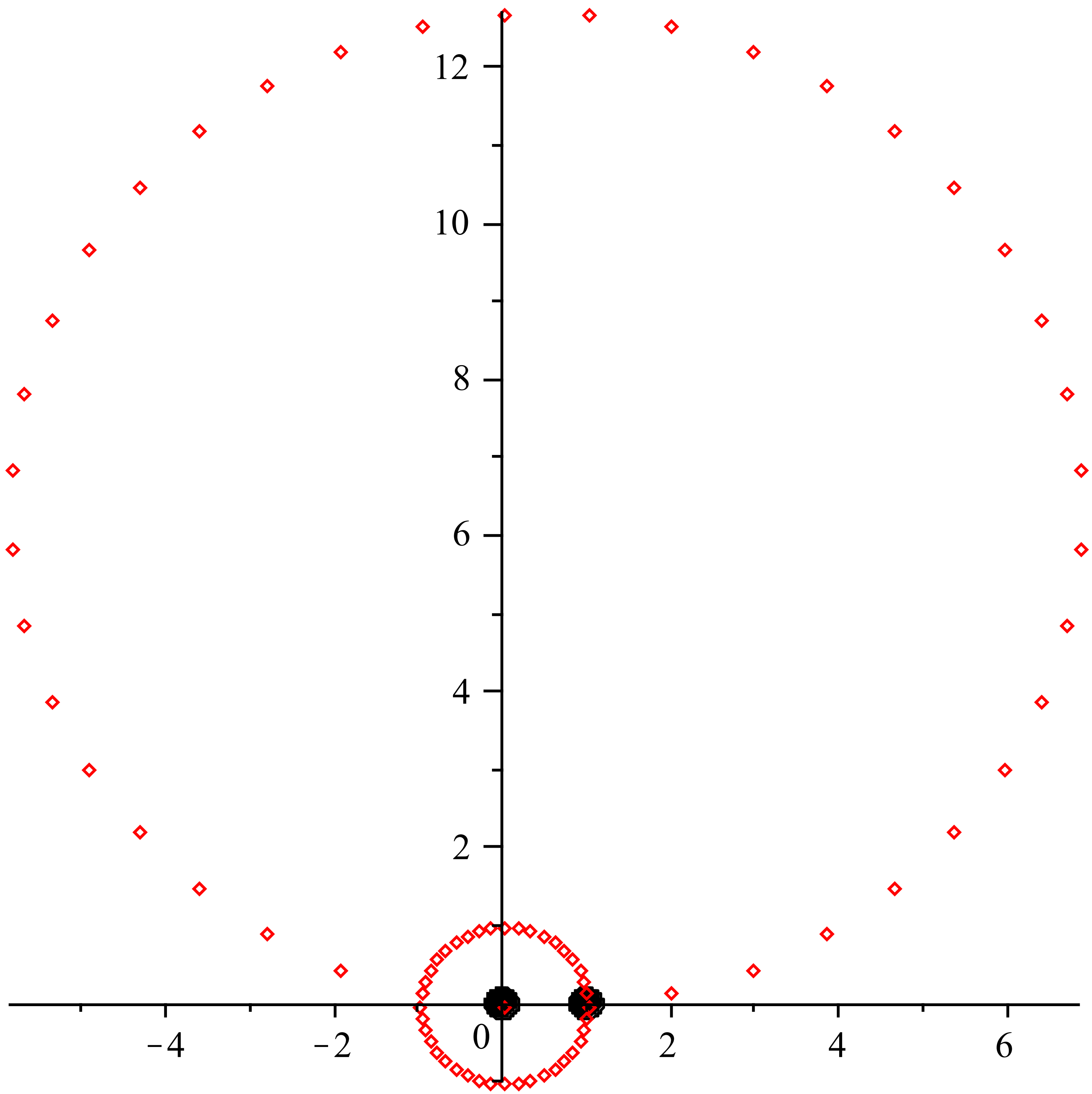

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The sum of two operators.

The operator T of the theorem of Augé will be $T = D + R$, where D, R satify :

$$R \circ D = R \qquad R^2 = 0$$

So, for each $n \geq 1$,

$$T^n = D^n + (I + D + \cdots + D^{n-1}) \circ R.$$

If X has a Schauder basis (e_n),

$$P : X \to \mathbb{C}^2 \qquad x = \sum_{n=1}^{+\infty} x_n e_n \qquad Px = (x_1, x_2)$$

Let $F = \Big\{(x_1, x_2) \in \mathbb{C}^2;\ |x_1| \geq |x_2|\Big\}$.

$$A(T) = \Big\{x \in X;\ \|T^n x\| \to \infty\Big\} = \Big\{x \in X;\ Px \notin F\}$$

$$R(T) = \Big\{x \in X;\ \liminf \|T^n x - x\| = 0\Big\} = \Big\{x \in X;\ Px \in F\Big\}$$

Wild sequences of linear forms on $\mathbb{C}^2$.

Proposition : Let $F \subset \mathbb{C}^2$.
Assume F is closed and union of linear subspaces.
There exists linear forms $f_k : \mathbb{C}^2 \to \mathbb{C}$ such that :

- for all $x \in F$, $\quad \liminf |f_k(x)| = 0$,
- for all $x \notin F$, $\quad \lim |f_k(x)| = +\infty$.

Example : $F = \Big\{(z_1, z_2) \in \mathbb{C}^2;\ |z_1| \geq |z_2|\Big\}$
F and $\mathbb{C}^2 \backslash F$ have non empty interior.

Let $\mathbb{D} := \{z \in \mathbb{C};\ |z| < 1\}$. There exists $(a_k) \subset \mathbb{D}$ such that :

$$\forall z \in \mathbb{D} \quad \exists p_k \leq k,\ p_k \to \infty \quad \text{such that} \quad |z - a_{p_k}| \leq \frac{\alpha}{\sqrt{k}}.$$

$f_k(z_1, z_2) = k^{1/4}(z_2 - a_k z_1)$. Note $\|f_k\| = O(k^{1/4})$.

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Let $\mathbb{D} := \{z \in \mathbb{C};\ |z| < 1\}$. There exists $(a_k) \subset \mathbb{D}$ such that :

$\forall z \in \mathbb{D} \quad \exists p_k \leq k,\ p_k \to \infty$ such that $|z - a_{p_k}| \leq \frac{\alpha}{\sqrt{k}}$.

$f_k(z_1, z_2) = k^{1/4}(z_2 - a_k z_1)$. Note $\|f_k\| = O(k^{1/4})$.

The diagonal operator D.

If $m_k = k!$, then $m_k | m_{k+1}$ and $m_p \sum_{k>p} \frac{\|f_k\|}{m_k} \to 0$ as $p \to \infty$.

$\lambda_1 = \lambda_2 = 1$ and for $k \geq 3$, $\lambda_k = e^{i\pi/m_{k+1}}$.

$$x = \sum_{n=1}^{+\infty} x_n e_n \qquad Dx = \sum_{n=1}^{+\infty} \lambda_n x_n e_n$$

where (e_n) is a Schauder basis of X.

If X has no Schauder basis, work with a Markushevic basis.

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- D is bounded $\qquad$ $D - I$ compact
- for all $x \in X$, $\lim_n D^{2m_n} x = x$, $\qquad$ $\lambda_k^{2m_n} = 1$ if $n > k$

The operator R.

$$P : X \to \mathbb{C}^2 \qquad x = \sum_{n=1}^{+\infty} x_n e_n \qquad Px = (x_1, x_2)$$

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- $(I + D + \cdots + D^{n-1}) e_k = \lambda_{k,n} e_k = \left(\sum_{\ell=0}^{n-1} \lambda_k^\ell \right) e_k.$

$\lambda_{k,2m_p} = 0$ if $p \geq k$	$\lvert\lambda_{k,n}\rvert \geq \frac{2}{\pi} n$ if $n \leq m_k$	$\lvert\lambda_{k,n}\rvert \leq n$ for all n

Construction of T.

$$P : X \to \mathbb{C}^2 \qquad x = \sum_{n=1}^{+\infty} x_n e_n \qquad Px = (x_1, x_2)$$

$$Tx = Dx + Rx = \sum_{k=1}^{+\infty} \lambda_k x_k e_k + \sum_{k=3}^{+\infty} \frac{f_k(Px)}{m_k} e_k$$

- $T - I = (D - I) + R$ is compact,
- $T^n x = D^n x + \sum_{k=3}^{+\infty} \frac{\lambda_{k,n} f_k(Px)}{m_k} e_k$ where $\lambda_{k,n} = \sum_{\ell=0}^{n-1} \lambda_k^\ell$

$$A(T) = \Big\{x \in X;\ \|T^n x\| \to \infty\Big\} = \Big\{x \in X;\ Px \notin F\Big\}$$

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Enough to show :

$Px \notin F \Rightarrow \|T^n x\| \to +\infty$ and $Px \in F \Rightarrow \liminf \|T^n x - x\| = 0$

If $Px \notin F$ and $m_k \leq n \leq m_{k+1}$, then $|\lambda_{k,n}| \geq \frac{2}{\pi} n \geq \frac{2}{\pi} m_k$, so

$\left|\frac{\lambda_{k,n} f_k(Px)}{m_k}\right| \geq \frac{2}{\pi} |f_k(Px)| \to \infty$ as $k \to \infty$. Hence $\|T^n x\| \to +\infty$

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Construction of T.

$$T^n x = D^n x + R_n x \quad \text{where} \quad R_n x = \sum_{k=3}^{+\infty} \frac{\lambda_{k,n} f_k(Px)}{m_k} e_k$$

$$\liminf \|T^n x - x\| \leq \liminf \|T^{2m_p} x - x\| \leq \liminf \|R_{2m_p} x\|$$

because $T^{2m_p}x - x = \left(D^{2m_p}x - x\right) + R_{2m_p}x$ and $\|D^{2m_p}x - x\| \to 0$.

If $k < p$, $\quad \lambda_{k,2m_p} = 0$.

If $k = p$, $\quad \left|\frac{\lambda_{p,2m_p} f_p(Px)}{m_p}\right| \leq 2|f_p(Px)|$.

If $k > p$, $\quad \left|\frac{\lambda_{k,2m_p} f_k(Px)}{m_k}\right| \leq \frac{2m_p}{m_k}\|f_k(Px)\| \leq 2m_p \frac{\|f_k\|}{m_k} \cdot \|Px\|$

So $\quad \liminf \|R_{2m_p} x\| \leq 2 \liminf |f_p(Px)|$.

Finally, if $Px \in F$, then $\quad \liminf \|T^n x - x\| = 0$.

The operator U.

Let $H = \bigoplus_2(\mathbb{C}^{2m_k}, \|\cdot\|_\infty)$ $U \in \mathcal{L}(H)$ restricted to $\mathbb{C}^{2m_k}$ satisfies

$$U(e_n) = \alpha_k e_{n+1}$$

whenever $1 \leq n < 2m_k$ where $\alpha_k^{2m_k-1} = k$ and

$$U(e_{2m_k}) = e_1/k.$$

For all $x \in X$, $\lim_k U^{2m_k}x = x$ and U non invertible.

(recall, we also have $\lim_k S^{2m_k}x = x$).

Theorem (S. Tapia)
If $T' \in \mathcal{L}(X \times H)$ is defined by $T'(x,y) = (Tx, Uy)$, then

- $\left\{R(T'), A(T')\right\}$ is a partition of $X \times H$,
- $R(T') = R(T) \times H$ and $A(T') = A(T) \times H$ have non empty interior.
- T' is not invertible.