Hadamard Inverse Function Theorem Proved by Variational Analysis

Milen Ivanov and Nadia Zlateva

International Conference: Challenges and advances in modern variational analysis

Celebrating Professor Jean-Paul Penot

14-15 March 2023, Limoges, France

The classical example $(x, y) \to e^x(\cos y, \sin y)$ shows that – except in dimension one – the derivative may be everywhere invertible while the function itself is invertible only locally.

Probably the historically first sufficient condition for global invertibility is given by J. S. Hadamard, and we will see it below.

An excellent overview – both from research and educational perspective – of this topic is given in

R. Plastock, Homeomorphisms between Banach spaces, Trans. Amer. Math. Soc., 200, 1974, 169–183.

We present a new proof of Hadamard Inverse Function Theorem based on an idea by I. Ekeland and E. Séré we found in the presentation draft

I. Ekeland and E. Séré, A local surjection theorem, 2017, https://project.inria.fr/brenier60/files/2011/12/Brenier.pdf

This idea allows obtaining a continuous right inverse to f on any compact set. The necessary key Proposition is proved by methods of Variational Analysis in the flavour of the monographs:

J.-P. Penot, *Calculus Without Derivatives*, Graduate Texts in Mathematics, vol. 266, Springer, 2013, ISBN: ISBN 978-1-4614-4537-1

A. L. Dontchev and R. T. Rockafellar, *Implicit Functions and Solution Mappings: A View from Variational Analysis*, Series in Operations Research and Financial Engineering, Springer, 2014, ISBN: 978-1-4939-1037-3

A. Ioffe, *Variational Analysis of Regular Mappings: Theory and Applications*, Springer Monographs in Mathematics, 2017, ISBN: 978-3-319-64277-2

Asen L. Dontchev, *Lectures on Variational Analysis*, Applied Mathematical Sciences Series, vol. 205, Springer, 2021, ISBN: 978-3-030-79910-6

Preliminaries

We work in a Banach space $(X, \|\cdot\|)$. Recall that the function

$$f : X \to X$$

is called *Fréchet differentiable at* $x \in X$ if there is a bounded linear operator $f'(x) : X \to X$ such that

$$\lim_{\|h\| \to 0} \frac{f(x+h) - f(x) - f'(x)h}{\|h\|} = 0,$$

and f is called *smooth*, denoted $f \in C^1$, if the mapping

$$x \to f'(x)$$

is norm-to-norm continuous.

Theorem (Hadamard). Let $f \in C^1$, $f'(x)$ be invertible for all x and satisfying

$$\|[f'(x)]^{-1}\| \leq M, \quad \forall x \in X,$$

for some $M > 0$.
Then f is C^1 invertible on X.
In other words, there is $g \in C^1$ such that

$$g(f(x)) = f(g(x)) = x, \quad \forall x \in X.$$

The plan is:

- recall some known facts;
- use them to prove the key Proposition;
- complete the proof of the Theorem.

Local Inverse Function Theorem.

Let $f \in C^1$ and let $f'(x_0)$ be invertible. Then there are $\varepsilon, \delta > 0$ such that for each y such that

$$\|y - f(x_0)\| < \varepsilon$$

there is unique $x =: g(y)$ such that $\|x - x_0\| < \delta$ and

$$f(x) = y.$$

Moreover, $g \in C^1$ and

$$g'(f(x_0)) = [f'(x_0)]^{-1}.$$

The following statement is also well-known.

Lemma. Let f be C^1. Let $K \subset X$ be compact and let $r > 0$. Then $f(x+th) = f(x) + tf'(x)h + o(t)$ uniformly on $x \in K$ and $h \in rB_X$. More precisely, there is $\alpha(t) \to 0$ as $t \to 0$ such that

$$\sup\{\|f(x+th) - f(x) - tf'(x)h\| : x \in K,\ h \in rB_X\} \leq \alpha(t)t.$$

Next result is a precursor to Ekeland Variational Principle and can be found in Chapter 5, Section 1 of

J.-P. Aubin and I. Ekeland, *Applied nonlinear analysis*, John Wiley & Sons, New York, 1984, ISBN: 0-486-45324-3

Of course, it easily follows from EVP itself, and it is a variant of the so-called in the book of A. Ioffe "Basic Lemma". Here we present a proof based on what is called in the book of A. Ioffe "simple iteration".

Basic Lemma. Let $\mu : X \to \mathbb{R}^+ \cup \{\infty\}$ be lower semicontinuous and such that for some $r > 0$

$$\forall x : \; 0 < \mu(x) < \infty \Rightarrow \exists y : \; \mu(y) < \mu(x) - r\|y - x\|.$$

Then for each $x \in \operatorname{dom} \mu$ there is $y \in X$ such that

$$\mu(y) = 0 \text{ and } r\|y - x\| \leq \mu(x).$$

Proof of the Basic Lemma

Fix $x_0 \in \operatorname{dom} \mu$ such that $\mu(x_0) > 0$. Let $x_1, x_2, \ldots, x_n$ be already chosen in the following way. Set

$$\nu_n := \sup\{\|x - x_n\| : \ \mu(x) < \mu(x_n) - r\|x - x_n\|\}.$$

We are given that the set of x satisfying the inequality is nonempty, so $\nu_n > 0$. Also, since $\mu \geq 0$, we have that $\nu_n \leq \mu(x_n)/r < \infty$. Choose a x_{n+1} such that

$$\mu(x_{n+1}) < \mu(x_n) - r\|x_{n+1} - x_n\| \text{ and } \|x_{n+1} - x_n\| > \nu_n/2.$$

Note that

$$\|x_{n+1} - x_0\| \leq \sum_{i=0}^{n} \|x_{i+1} - x_i\| \leq \sum_{i=0}^{n} (\mu(x_i) - \mu(x_{i+1}))/r \leq \mu(x_0)/r.$$

If $\mu(x_{n+1}) = 0$, we are done. If not, we continue by induction.

If we would end up with an infinite sequence $(x_n)_0^\infty$, then

$$\sum_{i=0}^{\infty} \|x_{i+1} - x_i\| \leq \mu(x_0)/r,$$

so $x_n \to \bar{x}$ as $n \to \infty$, and $\|\bar{x} - x_0\| \leq \mu(x_0)/r$.
From $\|x_{n+1} - x_n\| > \nu_n/2$, it follows that $\nu_n \to 0$.

If $\mu(\bar{x}) > 0$, then we can find $\bar{y}$ such that

$$\mu(\bar{y}) < \mu(\bar{x}) - r\|\bar{y} - \bar{x}\|. \tag{1}$$

Since μ is lower semicontinuous, we will have for all n large enough $\mu(\bar{y}) < \mu(x_n) - r\|\bar{y} - x_n\|$. As by definition

$$\nu_n = \sup\{\|x - x_n\| : \ \mu(x) < \mu(x_n) - r\|x - x_n\|\}$$

we will have that $\nu_n \geq \|\bar{y} - x_n\|$ for all n large enough.
As $\nu_n \to 0$, we get that $\bar{y} = \bar{x}$ which contradicts (1).
So, $\mu(\bar{x}) = 0$ and we are done. $\square$

Right inverse à la Ekeland & Séré

The following is what distinguishes our proof of Hadamard Theorem.

Proposition. Let $f \in C^1$, $f'(x)$ be invertible for all x and let for some $M > 0$

$$\|[f'(x)]^{-1}\| \leq M, \quad \forall x \in X.$$

Let $K \subset X$ be compact. Then f has a continuous right inverse on K, that is, there is a continuous $g : K \to X$ such that

$$f(g(x)) = x, \quad \forall x \in K.$$

Moreover, if $f(0) = 0 \in K$ then there is a continuous right inverse g of f on K that satisfies

$$g(0) = 0.$$

Proof of the Proposition

Let $C(K, X)$ be the space of all continuous functions from K to X. It is clear that when equipped with the norm

$$\|g\|_\infty := \max_{y \in K} \|g(y)\|$$

it is a Banach space. Consider the function

$$\mu : C(K, X) \to \mathbb{R}^+$$

which measures how much a given function g differs from a right inverse of f, i.e.

$$\mu(g) := \max_{y \in K} \|f(g(y)) - y\|.$$

It is clear that μ is lower semicontinuous.
Proposition claims that there exists g such that $\mu(g) = 0$.

We will check the condition of Basic Lemma. To this end, fix $\hat{g} \in C(K, X)$ such that

$$\mu(\hat{g}) > 0.$$

Set $u : K \to X$ as

$$u(y) := y - f(\hat{g}(y)).$$

By definition,

$$\mu(\hat{g}) = \|u\|_\infty.$$

As $\mu(\hat{g}) > 0$, u is not identically equal to zero. Put

$$w(y) := [f'(\hat{g}(y))]^{-1} u(y), \quad \forall y \in K.$$

Since $x \to f'(x)$ is continuous, it follows that

$$y \to [f'(\hat{g}(y))]^{-1}$$

is norm-to-norm continuous, so $w \in C(K, X)$.

Therefore, for $t > 0$

$$g_t := \hat{g} + tw \in C(K, X),$$

and form Hadamard condition it follows that
$\|w\|_\infty \leq M\|u\|_\infty = M\mu(\hat{g})$.
Our next aim is to estimate $\mu(g_t)$. By definition

$$\mu(g_t) := \max_{y \in K} \|f(g_t(y)) - y\|.$$

For $y \in K$ define $\varphi_y : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi_y(t) := \|f(g_t(y)) - y\|,$$

hence

$$\mu(g_t) = \max_{y \in K} \varphi_y(t).$$

Because the set $\hat{g}(K)$ is compact and the set $w(K)$ is bounded, from the Lemma it follows that

$$\max_{y\in K} \|f(\hat{g}(y)+tw(y)) - f(\hat{g}(y)) - tf'(\hat{g}(y))w(y)\| = \alpha(t)t,$$

where $\alpha(t) \to 0$ as $t \to 0$. But

$$f'(\hat{g}(y))w(y) = f'(\hat{g}(y))[f'(\hat{g}(y))]^{-1}u(y) = u(y),$$

so

$$\|f(g_t(y)) - f(\hat{g}(y)) - tu(y)\|_\infty = \alpha(t)t.$$

Therefore, for any $y \in K$

$$\begin{aligned}
\varphi_y(t) &= \|f(g_t(y)) - y\| \\
&\le \|f(\hat{g}(y)) + tu(y) - y\| + \|f(g_t(y)) - f(\hat{g}(y)) - tu(y)\| \\
&\le \|(t-1)u(y)\| + \alpha(t)t.
\end{aligned}$$

Since $\varphi_y(0) = \|u(y)\|$ we have that for small t

$$\varphi_y(t) \leq (1-t)\varphi_y(0) + \alpha(t)t.$$

Taking a maximum over $y \in K$, and then using that $\mu(g_t) = \max_{y\in K} \varphi_y(t)$, we get

$$\mu(g_t) \leq (1-t)\mu(g_0) + \alpha(t)t, \quad \text{or}$$

$$\mu(\hat{g} + tw) \leq \mu(\hat{g}) - t\mu(\hat{g}) + \alpha(t)t.$$

Since $\mu(\hat{g}) > 0$, for some $\delta > 0$ we then have $|\alpha(t)| < \mu(\hat{g})/2$ for $t \in (0, \delta)$. So,

$$\mu(\hat{g} + tw) < \mu(\hat{g}) - (t/2)\mu(\hat{g}), \quad \forall t \in (0, \delta).$$

From $\|w\|_\infty \leq M\mu(\hat{g})$, which is $\mu(\hat{g}) \geq (1/M)\|w\|_\infty$, we get

$$\mu(\hat{g} + tw) < \mu(\hat{g}) - (1/2M)\|tw\|_\infty, \quad \forall t \in (0, \delta).$$

So, we can apply Basic Lemma with $r = 1/2M$, $x = \hat{g}$ and $y = \hat{g} + (\delta/2)w$, to conclude that μ vanishes somewhere.

If $f(0) = 0 \in K$ then we can modify the above reasoning by considering instead of $C(K, X)$ the Banach space of continuous $g : K \to X$ such that $g(0) = 0$. It is clear that in this case $u(0) = w(0) = 0$ and everything else works in the same way. $\square$

Theorem (Hadamard)

Let $f \in C^1$, $f'(x)$ be invertible for all x and satisfying

$$\|[f'(x)]^{-1}\| \leq M, \quad \forall x \in X,$$

for some $M > 0$. Then f is C^1 invertible on X, i.e. there is $g \in C^1$ such that

$$g(f(x)) = f(g(x)) = x, \quad \forall x \in X.$$

Proof. It is enough to show that f is bijective.

Let $y \in X$ be arbitrary and set $K = \{y\}$. From the Proposition it follows that there is $x = g(y)$ such that $f(x) = y$. So, f is surjective, i.e. $f(X) = X$.

Let $a, b \in X$ be such that $f(a) = f(b)$. By considering instead of f the function

$$x \to f(b - x) - f(b)$$

we can assume without loss of generality that $b = 0$ and $f(0) = 0$. Then $f(a) = 0$.

Set $K := f([0,a])$. As f is continuous, K is compact. From the Proposition there is a continuous

$$g : K \to X, \text{ such that } g(0) = 0 \text{ and } f(g(y)) = y, \quad \forall y \in K.$$

Consider

$$I := \{t \in [0,1] : \; g(f(ta)) = ta\}.$$

Obviously, $0 \in I$, because $g(0) = 0$. Due to the continuity of g and f, the set I is closed and, therefore, compact. Set

$$\bar{t} := \max\{t : t \in I\}.$$

Assume that $\bar{t} < 1$.

By the Local Inverse Function Theorem, applied at the point $\bar{t}a$, there are $\delta, \varepsilon > 0$ such that for each $y \in X$ such that

$$\|y - f(\bar{t}a)\| < \varepsilon$$

there is unique $x \in X$ such that $\|x - \bar{t}a\| < \delta$ and $f(x) = y$.

From the continuity of f, there is $\beta > 0$ such that for all $t \in (\bar{t}, \bar{t} + \beta) \subset (0, 1)$ we have $\|ta - \bar{t}a\| < \delta$, $\|f(ta) - f(\bar{t}a)\| < \varepsilon$.

Hence, for $t \in (\bar{t}, \bar{t} + \beta)$, ta is the only solution of $f(\cdot) = f(ta)$.

Since $f(g(y)) = y$ for all $y \in K$ and since $f(ta) \in K$, we have that $f(g(f(ta))) = f(ta)$.

From the uniqueness of the solution to $f(\cdot) = f(ta)$ we obtain that $g(f(ta)) = ta$ for all $t \in (\bar{t}, \bar{t} + \beta)$ which contradicts the definition of $\bar{t}$.

So, $\bar{t} = 1$ meaning that $g(f(a)) = a$. As $f(a) = 0$ and $g(0) = 0$, it follows that $a = 0$.

We have proved that if $f(a) = f(b)$ then $a = b$, so f is injective. $\square$

Results are announced in the preprint

arXiv:2202.09327 [math.FA] Milen Ivanov, Nadia Zlateva, Hadamard Inverse Function Theorem Proved by Variational Analysis, February 2022

Joyeux anniversaire, professeur Penot!