Revealed Preferences

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En hommage à Jean-Paul Penot

On the primal side

Here $K \subset \mathbb{R}^d_+$ is the convex set of goods ordered by $\prec$, $\preceq$ and $\simeq$.

Given $x, y, z \in K$,

1) Either $x \prec y$ (y strictly preferred to x), either $y \prec x$, either $x \simeq y$.

2) If $x_i \leq y_i$ for all i and $x \neq y$ then $x \prec y$.

3) Let $0 < t < 1$ then :

a) if $x \preceq y$ then $x \preceq x + t(y - x)$

b) if $x \prec y$ then $x \prec x + t(y - x)$

4) if $x \preceq y \preceq z$ then $x \preceq z$.

5) if $x \preceq y \prec z$ or $x \prec y \preceq z$ then $x \prec z$.

Set for all $x \in K$,

$S_x = \{y \in K : x \preceq y\}, S_x^s = \{y \in K : x \prec y\}$

The sets S_x and S_x^s are convex.

Assume now that the S_x are closed and the S_x^s are open, then $\exists\, u$ (Debreu) s.t.

$u(x) \leq u(y) \Longleftrightarrow x \preceq y$

$u(x) < u(y) \Longleftrightarrow x \prec y$.

u is strictly increasing and quasiconcave.

On the dual side

If $\pi \in \Pi = \mathbb{R}^d_+$ is the vector of the unitary prices of goods, the cost of $x \in K$ is $\pi^t x$.

If the budget of the consumer is $w > 0$, the best choices belong to

$$X(\pi, w) = \{x \in K : x \prec y \Longrightarrow \pi^t y > \pi^t x = w\}.$$

Set $p = \pi / w$, $X(\pi, w) = X(p, 1) = X(p)$.

$$X(p) = \arg\max_x [u(x) : p^t x \leq 1].$$

The correspondance X is called demand.

$$v(p) = \max_x [\, u(x) \ : \ p^t x \leq 1],$$

When all things work well

(Lau, Diewert, Crouzeix, Martinez-Legaz,....)

$$u(x) = \min_p [\, v(p) \, : \, p^t x \leq 1].$$

v is strictly decreasing, quasiconvex

$$x \in X(p) \Longleftrightarrow u(x) = v(p), p^t x = 1 \Longleftrightarrow p \in P(x)$$

$$\text{with } \; \mathrm{P(x) = \arg\min_p [\, v(p) : p^t x} \leq 1].$$

The revealed preferences problem (Samuelson, Houthakker, Hurwicz -Uzawa, ...) consists in building an indirect utility function v (or a direct utility function u) from the observations on X.

When $X \in \mathcal{C}^1$.

Does there exist v quasiconvex, differentiable so that $X(p)$ is colinear to $\nabla v(p)$?

An easily seen necessary condition is :

The matrix $X'(p)$ is psd on $[X(p)]^\perp$. (CN)

What about sufficiency ?

Case $n = 2$. Rather easy : $\dim([X(p)]^\perp) = 1$, (CN) is also sufficient (Samuelson 1950).

Case $n > 2$. Very hard.

The necessary and sufficient condition

$X'(p)$ is psd and symmetric on $[X(p)]^{\perp}$.

2 types of proofs :

Construction of the indifference curves

Crouzeix-Rapcsak, 2005, with a very “ hand-made” proof.

Penot-Hadjisavvas, 2015, with a more scholar proof based on the Frobenius theorem.

Symplectic geometry, Darboux theorem

Chiappori-Ekeland, 1999, exterior differential calculus methods.

Revealed preferences axioms

a) Let us place in the case where the demand X is associated to the utility u. Then,

$X(p) = \arg\max[u(x) : p^t x \leq 1]$,

$x \in X(p) \Longrightarrow p^t x = 1$,

$x \in X(p)$ and $p^t(y - x) \leq 0 \Longrightarrow u(y) \leq u(x)$,

$x \in X(p)$ and $u(y) > u(x) \Longrightarrow p^t(y - x) > 0$.

Let a family $\{(x_i, p_i)\}_{i=0}^{q+1} \subset$ graph(X) so that $(x_0, p_0) = (x_{q+1}, p_{q+1})$, $p_i^t(x_{i+1} - x_i) \leq 0 \, \forall \, i < q + 1$.

Then, $u(x_0) \geq u(x_1) \geq \cdots \geq u(x_q) \leq u_0$.

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either $u(x_0) = u(x_{q+1})$. Then, for all i, $u(x_i) = u(x_0)$ and $p_i^t(x_{i+1} - x_i) = 0 \ \forall\, i$.

or $u(x_{q+1}) = u(x_0) > u(x_q)$, $p_q^t(x_{q+1} - x_q) > 0$ and $\max_i p_i^t(x_{q+1} - x_q) > 0$.

Hence the introduction of the RP axiom :

For any family $\{(x_i, p_i)\}_{i=0}^{q+1} \subset$ graph(X) so that $(x_0, p_0) = (x_{q+1}, p_{q+1})$ then $\max_i p_i^t(x_{i+1} - x_i) \geq 0$ and, if $max = 0$, all $p_i^t(x_{i+1} - x_i) = 0$.

Variants : WARP, SARP, GARP, Samuelson, Houthakker, Varian, Afriat, ...

When mathematicians learn from economists

Pseudomonotone maps and Cyclically Pseudomonotone maps are nothing but that the Revealed Preference Axioms introduced by economists about 30 years before.

Recall that the revealed preferences problem consists in building a utility function u from X.

The Afriat's constructions

Given X cycl. ps.mon. and a finite family $\{(x_j, p_j)\}_{j\in J} \subset$ graph(X), find $\alpha_j, \beta_j > 0$ s.t.

$$\alpha_k \geq \alpha_j + \beta_j p_j^t (x_k - x_j) \quad \forall\, j, k \in J.$$

Existence $\iff$ X cycl. ps.mon. on J. Set

$$u_J(x) = \min_j \left[\alpha_j + \beta_j p_j^t (x - x_j)\right] \ \forall\, x,$$

u_J is concave, piecewise linear, $u_J(x_j) = \alpha_j$.

$$x_j \in \arg\max_y \left[\, u_J(y) \ : \ p_j^t (y - x_j) \leq 0\right] \ \forall\, j \in J.$$

α_j, β_j usually obtained in solving in a linear program, no unicity of u_J.

See also, Diewert, Fostel-Scarf-Todd, ...

Rescalarizations needed for comparisons

Set $e = (1, 1, \cdots, 1) \in \mathbb{R}^d$ and

$k_J(t) = u_J(te)$: k_J is concave, strict. $\nearrow$.

Next, take $\tilde{u}_J = [k_J]^{-1} \circ u_J$,

$\tilde{u}_J$ is pseudoconcave, $\tilde{u}_J(te) = t$,

$\tilde{u}_J(y) \geq \tilde{u}_J(x_j) \Longleftrightarrow p_j^t(y - x_j) \geq 0$.

Credibility of $x \preceq y$ when $\tilde{u}_J(x) \leq \tilde{u}_J(y)$?

If u is a normalized utility and X is the associate demand, how u is approximated by $\tilde{u}_J$? what happens when $J \nearrow$, when u is not concavifiable ?

Sandwich inequalities, the finite case

Let X, J as in the Afriat's construction.
Let $\mathcal{U}_J$ be the class of $\nearrow$ quasiconcave functions u on K such that $u(te) = t \;\; \forall t$ and

$$x_j \in \arg\max_y [\, u(y) \; : \; p_j^t(y - x_j) \leq 0] \;\; \forall j \in J.$$

Then, (Crouzeix-Keraghel-Rahmani)

$$\exists\, u_J^-, u_J^+ \in \mathcal{U}_J \quad \text{st} \quad \mathrm{u}_\mathrm{J}^- \leq \mathrm{u} \leq \mathrm{u}_\mathrm{J}^+ \;\; \forall\, \mathrm{u} \in \mathcal{U}_\mathrm{J}.$$

u_J^-, u_J^+ built via easy OR technics, competitive with Afriat's constructions.

$$J_1 \subset J_2 \Longrightarrow u_{J_1}^- \leq u_{J_2}^- \leq u_{J_2}^+ \leq u_{J_1}^+.$$

Sandwich inequalities, infinite case.

Let $\mathcal{U}$ be the class of $\nearrow$ quasiconcave functions u such that $u(te) = t \;\; \forall\, t$ and

$$X(p) \subset \arg\max_y \,[\, u(y) : \; p^t y \leq 1] \quad \forall\, p.$$

Set for all $x \in K$

$$J(x) = \left\{ y : \begin{array}{c} \exists\, k \in \mathbb{N}, \; x_0 = x, \, x_k = y, \\ (x_0, p_0), \cdots, (x_k, p_k) \in \text{graph(X)}, \\ p_i^t (x_{i+1} - x_i) \leq 0, \, \forall\, i = 0, \cdots, p-1 \end{array} \right\}.$$

Making of the two slices of the sandwich

Define

$u^-(x) = \text{sup}_t\,[\,t\,:\,te \in J(x)\,]$,

$u^+(x) = \text{inf}_t\,[\,t\,:\,x \in J(te)\,]$.

then (Crouzeix-Eberhard-Ralph), u^- and u^+ are quasiconcave, $\nearrow$, belong to $\mathcal{U}$ and

$$u^- \leq u \leq u^+ \quad \forall\, u \in \mathcal{U}.$$

It can be expected $u^- = u^+$, the existence and unicity of $\prec$ associated with a utility u

Very bad news

There is a counter-example (Crouzeix-Eberhard-Ralph) where X is cyclic pseudomonotone, maximal pseudomonotone, $u^- = u^+$ but these functions are not pseudoconcave.

In another counter-example, X is cyclic pseudomonotone, maximal pseudomonotone but $u^- \neq u^+$. This means that there are different orders sharing the same demand.

Cyclic pseudomontonicity together with maximality are not enough