Compactly locally uniformly convex functions[1]

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Motivation

Let $(X, \|\cdot\|)$ be a real Banach space; one says that:

X is *uniformly convex (rotund)* if

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x, y \in S_X\ :\ \|y - x\| \geq \varepsilon \Rightarrow \left\|\tfrac{1}{2}(x + y)\right\| \leq 1-\delta,$$

X is *locally uniformly convex (rotund)* if

$$\forall\, x \in S_X,\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, y \in S_X\ :\ \|y - x\| \geq \varepsilon \Rightarrow \left\|\tfrac{1}{2}(x + y)\right\| \leq 1 - \delta.$$

We may continue with

X is *locally uniformly convex (rotund) at* $x_0 \in S_X$ if

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x \in S_X\ :\ \|x - x_0\| \geq \varepsilon \Rightarrow \left\|\tfrac{1}{2}(x + x_0)\right\| \leq 1-\delta.$$

The sequential characterizations:

X is uniformly convex (UC) if and only if

$$\forall (x_n), (y_n) \subset S_X : \left\| \tfrac{1}{2}(x_n + y_n) \right\| \to 1 \Rightarrow \|x_n - y_n\| \to 0;$$

X is locally uniformly convex (LUR) if and only if

$$\forall x \in S_X,\ \forall (x_n) \subset S_X : \left\| \tfrac{1}{2}(x + x_n) \right\| \to 1 \Rightarrow \|x_n - x\| \to 0;$$

X is LUR at $x_0 \in S_X$ if and only if

$$\forall (x_n) \subset S_X : \left\| \tfrac{1}{2}(x_n + x_0) \right\| \to 1 \Rightarrow \ \|x_n - x_0\| \to 0.$$

These notions were extended to proper convex functions.

Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function.

"The functional $f(x)$ is called *uniformly convex* if there exists a function $\delta(r)$, $\delta(r) > 0$ for $r > 0$ (which can be assumed monotonic), such that $f((x+y)/2) \leq \frac{1}{2}(f(x)+f(y)) - \delta(\|x-y\|)$ for all x, y";[2] equivalently,

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x, y \in \operatorname{dom} f : \\ \|x-y\| \geq \varepsilon \ \Rightarrow\ f(\tfrac{1}{2}x + \tfrac{1}{2}y) \leq \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y) - \delta;$$

f is *locally uniformly convex* (LUC or LUR) if

$$\forall x \in \operatorname{dom} f,\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, y \in \operatorname{dom} f : \\ \|x-y\| \geq \varepsilon \ \Rightarrow\ f(\tfrac{1}{2}x + \tfrac{1}{2}y) \leq \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y) - \delta;$$

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[2] [LP66] E.S. Levitin, B.T. Polyak, *Convergence of minimizing of sequences in the conditional-extremum problem*, Dokl. Akad. Nauk SSSR 168 (1966), 997–1000.

f is *locally uniformly convex at* $x_0 \in \operatorname{dom} f$ if[3]

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x \in \operatorname{dom} f : \\ \|x - x_0\| \geq \varepsilon \ \Rightarrow\ f(\tfrac{1}{2}x + \tfrac{1}{2}x_0) \leq \tfrac{1}{2}f(x) + \tfrac{1}{2}f(x_0) - \delta.$$

We may add the strongest convexity notion (possibly, inspired by the parallelogram law):

"A functional $f(x)$ is said to be *strongly convex* if there exists a $\gamma > 0$ such that $f((x+y)/2) \leq \frac{1}{2}(f(x) + f(y)) - \frac{1}{4}\gamma \|x - y\|^2$ for all x, y".[4]

[3][Z83] C. Z., *On uniformly convex functions*, J. Math. Anal. Appl. 95 (1983), 344–374.

[4][P66] B.T. Polyak, *Existence theorems and convergence of minimizing of sequences in extremum problems with constraints*, Dokl. Akad. Nauk SSSR 166 (1966), 287–290.

Of course, each of the above convexity notions has sequential variants; for example,

f is *locally uniformly convex at* $x_0 \in \operatorname{dom} f$ if and only if[5]

$$\forall (x_n) \subset \operatorname{dom} f : \tfrac{1}{2} f(x_n) + \tfrac{1}{2} f(x_0) - f(\tfrac{1}{2} x_n + \tfrac{1}{2} x_0) \to 0 \Rightarrow x_n \to x_0;$$

f is *locally uniformly convex* if it is locally uniformly convex at each $x \in \operatorname{dom} f$.

In 1973, Vlasov[6] introduced the following notion:

X is *compactly locally uniformly convex (rotund)* if (x_n) has a convergent subsequence whenever $x, x_n \in X$, $\|x_n\| = \|x\| = 1$ and $\|x_n + x\| \to 2$.

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[5][BV11] J.M. Borwein, J.D. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counterexamples*, Cambridge University Press, 2011.

[6][V73] L.P. Vlasov, Approximative properties of sets in normed linear spaces, Russian Math. Surveys, 28 (1973), 1–66.

So, it is natural to extend this notion to convex functions; more precisely, we say that

the proper convex function $f : X \to \overline{\mathbb{R}}$ is *locally uniformly convex* or *rotund* (CLUR for short) *at* $x_0 \in \operatorname{dom} f$ if $(x_n) \subset \operatorname{dom} f$ has a convergent subsequence whenever $\frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f(\frac{1}{2}x_n + \frac{1}{2}x_0) \to 0$.

Our aim is to characterize the previous notion and to study its relation with other notions that correspond to geometric properties of Banach spaces.

The sets $C_\delta f(x_0)$, $A_\delta^X f(x_0)$, $A_\delta f(x_0)$ that appear in the statement of next result are defined later on, while $\alpha(S)$ is the Hausdorff index of non-compactness of $S \subset X$.

Our main result is the following:

Theorem

Let f be continuous function at $x_0 \in \operatorname{dom} f$. The following statements are equivalent.

(i) f is CLUR at x_0;

(ii) If $x_n \in C_{1/n}f(x_0)$ for every n, then there exist a subsequence (x_{n_k}) and $y_0 \in C_0 f(x_0)$ such that $x_{n_k} \to y_0$;

(iii) $\alpha(A^X_{1/n}f(x_0)) \to 0$;

(iv) $\alpha(\overline{A^X_{1/n}f(x_0)}) \to 0$;

(v) $\overline{A^X_0 f(x_0)}$ is compact and for every $\varepsilon > 0$ such that $\overline{A^X_{1/n}f(x_0)} \subseteq B_\varepsilon(A^X_0 f(x_0))$ for large n;

(vi) $\operatorname{cl}(A_0 f(x_0))$ is compact and for every $\varepsilon > 0$, there exists $n \geq 1$ such that $\operatorname{cl}(A_{1/n}f(x_0)) \subseteq B_\varepsilon(A_0 f(x_0))$ for large n;

(vii) $\alpha(\operatorname{cl}(A_{1/n}f(x_0))) \to 0$.

(viii) $\alpha(A_{1/n}f(x_0)) \to 0$.

Notations and preliminary results

- $(X, \|\cdot\|)$ is a Banach space, $(X^*, \|\cdot\|)$ its topological dual endowed with the dual norm, $(X^{**}, \|\cdot\|) := ((X^*, \|\cdot\|))^*$ the dual of X^*
- $\langle x, x^* \rangle := x^*(x)$ for $x \in X$ and $x^* \in X^*$
- $J_X : X \to X^{**}$ is the canonical isometry: $x^{**} := J_X(x) \in X^{**}$ is defined by $x^{**}(x^*) := \langle x, x^* \rangle$ for all $x^* \in X^*$; X is identified with $J_X(X) \subset X^{**}$, and so $x \in X$ and $A \subset X$ are oftenly identified with $J_X(x)$ and $J_X(A)$, respectively
- $\langle x^*, x \rangle := \langle x^*, J_X(x) \rangle$ for $x \in X$ and $x^* \in X^*$
- S_X, B_X, U_X the unit sphere, unit open ball, unit closed ball of X
- $w := \sigma(X, X^*)$ $(\subset \tau_{\|\cdot\|})$, $w^* := \sigma(X^*, X)$ $(\subset \sigma(X^*, X^{**}))$

• For $A \subset X$: $\mathrm{cl}_{w^*} A$ is the w^*-closure of A in X^{**}, $\overline{A}$ is norm-closure of A in X; $\operatorname{int} A$ is the norm-interior of A

• For $A \subset X$ or $A \subset X^{**}$, $\operatorname{cl} A$ is the norm-closure of A in X^{**}.

• $\mathcal{F}_E := \{F \subset E \mid F \text{ is finite}\}$, where E is a set.

• $\alpha(A) := \inf\{r > 0 \mid \exists E \in \mathcal{F}_X : A \subset E_r := E + rB_X\} \in [0, \infty]$ (with $\inf \emptyset := \infty$), the Hausdorff index of noncompactness of $\emptyset \neq A \subset X$

• $A_n \to^{H^+} A$ for $\emptyset \neq A, A_n \subset X$ $(n \geq 1)$ if $\forall r > 0$, $\exists n_r \geq 1$, $\forall n \geq n_r : A_n \subset A + rB_X$

Consider $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$

• $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$, the domain of f; f is proper if $\operatorname{dom} f \neq \emptyset$ and $f(x) \neq -\infty$ for every $x \in X$

- $\Lambda(X) := \{f : X \to \overline{\mathbb{R}} \mid f$ is proper and convex$\}$
- $\Gamma(X) := \{f \in \Lambda(X) \mid f$ is l.s.c.$\}$
- $f^* : X^* \to \overline{\mathbb{R}}$ is the conjugate of f;
$f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x) \mid x \in X\}$; f^* is convex and w^*-lsc
- $\partial_\varepsilon f(x_0) := \{x^* \in X^* \mid \forall x \in X : \langle x - x_0, x^*\rangle \leq f(x) - f(x_0) + \varepsilon\}$ if $f(x_0) \in \mathbb{R}$, $\varepsilon \in \mathbb{R}_+ := [0, \infty[$, $\partial_\varepsilon f(x_0) := \emptyset$ if $f(x_0) \notin \mathbb{R}$, the ε-subdifferential of f at x_0; $\partial_\varepsilon f(x_0)$ is w^*-closed and convex; $\partial f(x_0) := \partial_0 f(x_0)$
- $\partial_\varepsilon f^*(x_0^*) \subset X^{**}$; $\partial_\varepsilon^X f^*(x_0^*) := X \cap \partial_\varepsilon f^*(x_0^*)$
- $\iota_E : F \to \overline{\mathbb{R}}$, $\iota_E(u) := 0$ if $u \in E$, $\iota_E(u) := \infty$ if $u \in F \setminus E$ is the indicator function of $E \subset F$

The next result, concerning the Hausdorff index of non-compactness, is needed in some proofs. Notice that in the definition of $\alpha(A)$, as well as in statements and proofs, one may equivalently replace B_X by U_X.

Proposition HIN

([CSZ07, Props. 2.2, 2.3][a]) Let (C_n) be a sequence of nonempty closed subsets of X satisfying the condition $C_{n+1} \subseteq C_n$ for all n and let $C = \cap_{n=1}^{\infty} C_n$. Consider the following assertions:

(i) every sequence (x_n) with $x_n \in C_n, n \in \mathbb{N}$, has a convergent subsequence;

(ii) C is nonempty compact and $C_n \to^{H^+} C$.

(iii) there exists a compact set $E \subset X$ such that $C_n \to^{H^+} E$.

(iv) $\alpha(C_n) \to 0$ as $n \to \infty$.

(v) $\operatorname{diam}(C_n) \to 0$.

(vi) $\forall (x_n) \subset (C_n), \exists x \in X : x_n \to x$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftarrow$ (v) $\Leftrightarrow$ (vi). If C is a singleton, then (iv) $\Leftrightarrow$ (v).

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[a] [CSZ07] A. K. Chakrabarty, P. Shunmugaraj, C. Z., *Continuity properties for the subdifferential and ε-subdifferential of a convex function and its conjugate*, J. Convex Anal., 14 (2007), 479–514.

Characterizations of CLUR functions in terms of sub-differentials

In the sequel X is a Banach space and $f \in \Gamma(X)$.

Definition

(i) The function f is said to be *locally uniformly convex* or *locally uniformly rotund* (LUR, in short) *at* $x_0 \in \operatorname{dom} f$, if $x_n \to x_0$ whenever $x_n \in \operatorname{dom} f$ and $\frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f(\frac{1}{2}x_n + \frac{1}{2}x_0) \to 0$;

(ii) f is called *compactly locally uniformly convex* or *compactly locally uniformly rotund* (CLUR, in short) *at* $x_0 \in \operatorname{dom} f$, if (x_n) has a convergent subsequence whenever $(x_n) \subset \operatorname{dom} f$ and $\frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f(\frac{1}{2}x_n + \frac{1}{2}x_0) \to 0$;

(iii) f is LUR (resp., CLUR) if it is LUR (resp., CLUR) at all points of $\operatorname{dom} f$.

(iv) f is *strictly convex at* $x_0 \in \operatorname{dom} f$ if $f(\frac{1}{2}x + \frac{1}{2}x_0) < \frac{1}{2}f(x_0) + \frac{1}{2}f(x)$ for all $x \in \operatorname{dom} f \setminus \{x_0\}$.

It follows easily that f is strictly convex iff f is strictly convex at all points of $\operatorname{dom} f$.

For $x_0 \in \operatorname{dom} f$ and $\delta \in \mathbb{R}_+$ consider the set

$$C_\delta f(x_0) := \{y \in \operatorname{dom} f : f(\tfrac{1}{2}x_0+\tfrac{1}{2}y) \geq \tfrac{1}{2}f(x_0)+\tfrac{1}{2}f(y)-\delta\} \ \ (\ni x_0);$$

one has

$$f \text{ is strictly convex at } x_0 \quad \Leftrightarrow \quad C_0 f(x_0) = \{x_0\}.$$

Fact A

Assume that f is continuous at $x_0 \in \operatorname{dom} f$ and consider $(\delta_n)_{n\geq 1} \subset \mathbb{R}_+$ with $\delta_n \to \delta$ and $x_n \in C_{\delta_n} f(x_0)$ for $n \geq 1$. If $x_n \to y$, then $y \in C_\delta f(x_0)$. Consequently, $C_\delta f(x_0)$ is closed for every $\delta \in \mathbb{R}_+$. Moreover, if $\delta = 0$, then $f(x_n) \to f(y) = 2f(\frac{1}{2}x_0 + \frac{1}{2}y) - f(x_0)$.

Fact B

Having $x_0, x_1 \in \operatorname{dom} f$, set $x_\lambda := (1-\lambda)x_0 + \lambda x_1$.

(i) $x^* \in \partial f(x_0) \Leftrightarrow x_0 \in \arg\min(f - x^*)$;

(ii) if $x^* \in \partial f(x_0) \cap \partial f(x_1)$, then $x^* \in \partial f(x_\lambda)$ and $f(x_\lambda) = (1-\lambda)f(x_0) + \lambda f(x_1)$ for every $\lambda \in [0,1]$;

(iii) if $\lambda_0 \in \,]0,1[$ is such that $f(x_{\lambda_0}) = (1-\lambda_0)f(x_0) + \lambda_0 f(x_1)$, then $\partial f(x_\lambda) = \partial f(x_0) \cap \partial f(x_1)$ for all $\lambda \in \,]0,1[$.

Fact C

Let $x_0 \in \operatorname{dom} f$, and consider the following conditions:

(i) f is strictly convex at x_0;

(ii) $\partial f(x) \cap \partial f(x_0) = \emptyset$ for all $x \in \operatorname{dom} f \setminus \{x_0\}$;

(iii) $\partial f(x) \cap \partial f(x_0) = \emptyset$ for all $x \in \operatorname{int}(\operatorname{dom} f) \setminus \{x_0\}$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii); moreover, if f is continuous at x_0, then (iii) $\Rightarrow$ (i).

Recall that $\partial_\delta f^*(x^*) \subset X^{**}$ and $\partial_\delta^X f^*(x^*) := X \cap \partial_\delta f^*(x^*)$ for $x^* \in \operatorname{dom} f^*$ and $\delta \geq 0$, X being identified with $J_X(X)$. Set

$$A_\delta^X f(x_0) := \cup\{\partial_\delta^X f^*(x^*) \mid x^* \in \partial_\delta f(x_0) \cap \operatorname{Im} \partial f\};$$

because $\partial_0 f(x_0) = \partial f(x_0)$, one obviously has

$$\partial f(x_0) \neq \emptyset \Leftrightarrow x_0 \in A_0^X f(x_0) \Leftrightarrow A_0^X f(x_0) \neq \emptyset.$$

Fact D

Let $x_0 \in \operatorname{dom} f$ and $\delta \geq 0$. TFAH:

(i) $A_\delta^X f(x_0) \subset C_\delta f(x_0)$;

(ii) Assume that f is continuous at $x_0 \in \operatorname{dom} f$. Then $C_{\delta/2} f(x_0) \subset A_\delta^X f(x_0)$; consequently

$$C_{\delta/2} f(x_0) \subset A_\delta^X f(x_0) \subset \operatorname{cl} A_\delta^X f(x_0) \subset C_\delta f(x_0),$$
$$C_0 f(x_0) = A_0^X f(x_0) = \operatorname{cl} A_0^X f(x_0),$$

Because $f \in \Gamma(X)$, for $x^* \in \operatorname{dom} f^*$, also $\partial_\delta^X f^*(x^*)$ is closed for $\delta \geq 0$ and nonempty for $\delta > 0$. Clearly, $\partial_\delta f^*(x^*) \subset \partial_{\delta'} f^*(x^*)$ for $0 \leq \delta < \delta'$, and similarly $C_\delta f(x_0) \subset C_{\delta'} f(x_0)$ for $x_0 \in \operatorname{dom} f$. It follows that for every sequence $(\delta_n)_{n\geq 1} \subset \mathbb{P} := (0, \infty)$ with $\delta_n \to 0$, one has

$$\lim_{0<\delta\to 0} \alpha\left(\partial_\delta f^*(x^*)\right) = 0 \Leftrightarrow \alpha\left(\partial_{\delta_n} f^*(x^*)\right) \to 0,$$

$$\lim_{0<\delta\to 0} \alpha\left(\partial_\delta^X f^*(x^*)\right) = 0 \Leftrightarrow \alpha\left(\partial_{\delta_n}^X f^*(x^*)\right) \to 0,$$

$$\lim_{0<\delta\to 0} \alpha\left(C_\delta f(x_0)\right) = 0 \Leftrightarrow \alpha\left(C_{\delta_n} f(x_0)\right) \to 0,$$

$$\partial f^*(x^*) = \cap_{n\geq 1} \partial_{\delta_n} f^*(x^*),$$

$$\partial^X f^*(x^*) = \cap_{n\geq 1} \partial_{\delta_n}^X f^*(x^*),$$

$$C_0 f(x_0) = \cap_{n\geq 1} C_{\delta_n} f(x_0)$$

The next result is strongly related to Theorem 5.19 from [CSZ07].

Fact E

Let $x^* \in \operatorname{dom} f^*$, and consider the following assertions:

(i) $\lim_{0<\delta\to 0} \alpha\left(\partial_\delta^X f^*(x^*)\right) = 0$;

(ii) $\lim_{0<\delta\to 0} \alpha\left(\partial_\delta f^*(x^*)\right) = 0$;

(iii) $\partial^X f^*(x^*)$ is nonempty, compact and $\partial_{1/n}^X f^*(x^*) \to^{H^+} \partial^X f^*(x^*)$;

(iv) $\partial f^*(x^*)$ is nonempty, compact and $\partial_{1/n} f^*(x^*) \to^{H^+} \partial f^*(x^*)$ in X^{**};

(v) $\partial f^*(x^*)$ is a nonempty compact subset of X and $\partial_{1/n} f^*(x^*) \to^{H^+} \partial f^*(x^*)$ (in X^{**});

(vi) $\partial f^*(x^*)$ is a nonempty compact subset of X.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (vi).

Concerning the proof, observe that: (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) follow immediately from the equivalence (i) $\Leftrightarrow$ (iv) of Proposition HIN because the mapping $0 < \delta \mapsto \partial_\delta f^*(x^*)$ is increasing; (ii) $\Rightarrow$ (i) is obvious because $\partial_\delta^X f^*(x^*) \subset \partial_\delta f^*(x^*)$ for $\delta \geq 0$; (v) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (vi) are obvious.

The most involved is the implication (i) $\Rightarrow$ (v). For this one apply [CSZ07, Prop. 4.2] which states that for $f \in \Gamma(X)$, $x^* \in \operatorname{dom} f^*$ and $\varepsilon > 0$ one has that $\partial_\varepsilon f^*(x^*) = \operatorname{cl}_{w^*} \partial_\varepsilon^X f^*(x^*)$.

Notice that the equivalence of conditions (i), (ii) and (iii) from Fact E are established in [CSZ07, Th. 5.19] under the supplementary hypothesis that $x^* \in \operatorname{int}(\operatorname{dom} f^*)$.

The following result is a counterpart of [SV18, Th. 3.3];[7] as already said, it is our main result.

[7][SV18] P. Shunmugaraj, V. Thota, *Some geometric and proximinality properties in Banach spaces*, J. Convex Anal., 25 (2018), 1139-1158.

Theorem M

Let f be continuous function at $x_0 \in \operatorname{dom} f$. The following statements are equivalent.

(i) f is CLUR at x_0;

(ii) If $x_n \in C_{1/n}f(x_0)$ for every n, then there exist a subsequence (x_{n_k}) and $y_0 \in C_0 f(x_0)$ such that $x_{n_k} \to y_0$;

(iii) $\alpha(A^X_{1/n}f(x_0)) \to 0$;

(iv) $\alpha(\overline{A^X_{1/n}f(x_0)}) \to 0$;

(v) $A^X_0 f(x_0)$ is compact and $\overline{A^X_{1/n}f(x_0)} \to^{H^+} A^X_0 f(x_0)$;

(vi) $\operatorname{cl}(A_0 f(x_0))$ is compact and $\operatorname{cl}(A_{1/n}f(x_0)) \to^{H^+} A_0 f(x_0)$;

(vii) $\alpha(\operatorname{cl}(A_{1/n}f(x_0))) \to 0$.

(viii) $\alpha(A_{1/n}f(x_0)) \to 0$.

Sketch of the proof (in which we omit $f(x_0)$):

First observe that for every sequence $(\delta_n) \subset \mathbb{P}$ with $\delta_n \to 0$, one has

$$\begin{aligned} f \text{ is CLUR at } x_0 &\Leftrightarrow [\forall (x_n) \subseteq (C_{\delta_n}) : \exists (x_{n_k}) \to x \in X] \\ &\Leftrightarrow \forall (x_n) \subseteq (C_{1/n}) : \exists (x_{n_k}) \to x \in X, \end{aligned}$$

$$\begin{aligned} C_0 &= \cap_{n\geq 1} C_{1/n} = \cap_{n\geq 1} C_{\delta_n} \\ &= \cap_{n\geq 1} A^X_{1/n} = \cap_{n\geq 1} A^X_{\delta_n} = \cap_{n\geq 1} \overline{A^X_{1/n}} = \cap_{n\geq 1} \overline{A^X_{\delta_n}} = A^X_0. \end{aligned}$$

(i) $\Leftrightarrow$ (ii) is nothing else than the the first equivalence above.

(iii) $\Leftrightarrow$ (iv) and (vii) $\Leftrightarrow$ (iii) because $\alpha(A) = \alpha(\overline{A})$ in a metric space;

clearly, (ii) $\Leftrightarrow$ (ii') by (i) $\Leftrightarrow$ (iv) in Proposition HIN, where

(ii') $\alpha(C_{1/n}) \to 0$;

(ii') $\Leftrightarrow$ (iii) by Fact D;

(iv) $\Leftrightarrow$ (v) by (i) $\Leftrightarrow$ (ii) in Proposition HIN and the equalities $C_0 = \cap_{n\geq 1} C_{1/n} = \ldots$ above;

(viii) $\Rightarrow$ (iii) because $A_\delta^X \subseteq A_\delta$ for $\delta \in \mathbb{R}_+$.

(v) $\Rightarrow$ (vi) Fix $\varepsilon > 0$; by hypothesis, there exists $n_0 \in \mathbb{N}^*$ such that

$$\operatorname{cl} A_\delta^X \subseteq A_0^X + \varepsilon U_X \subseteq A_0^X + \varepsilon U_{X^{**}},$$

where $\delta := 1/n_0 > 0$. Consider $y^{**} \in A_\delta$; then there exists $x^* \in \partial_\delta f(x_0) \cap \operatorname{Im} \partial f$ such that $y^{**} \in \partial_\delta f^*(x^*)$.
One continues using [CSZ07, Prop. 4.2] as in the proof of Fact E, one gets $\operatorname{cl} A_0 \subset A_0^X \subset A_0$, and so $A_0 = A_0^X = \cap_{n\geq 1} \operatorname{cl} A_{1/n}$.

(vi) $\Rightarrow$ (vii) Because $\operatorname{cl}(A_{1/n} f(x_0)) \to^{H^+} A_0 f(x_0)$, one has $\operatorname{cl}(A_{1/n} f(x_0)) \to^{H^+} \operatorname{cl}(A_0 f(x_0))$; as $\operatorname{cl}(A_0 f(x_0))$ is compact, one has also $\operatorname{cl}(A_{1/n} f(x_0)) \to^{H^+} \operatorname{cl}(A_0 f(x_0))$. One applies now the implication (iii) $\Rightarrow$ (iv) from Proposition HIN

From Theorem M one obtains immediately the next corollary

Corollary F

Let f be be continuous at $x_0 \in \operatorname{dom} f$.

(i) If f is CLUR at x_0 then $A_0 f(x_0) = A_0^X f(x_0)$ and so $A_0 f(x_0)$ is non-empty and compact.

(ii) f is LUR at $x_0 \Leftrightarrow$ [f is CLUR at x_0 and $C_0 = \{x_0\}$]
$\Leftrightarrow$ f is CLUR at x_0 & f is strictly convex at x_0].

Definition

Let $x_0 \in \operatorname{dom} f$; f is said to be:

(i) *strongly convex at* x_0, if $\partial f(x_0) \neq \emptyset$ and (x_n) converges whenever $(f - x^*)(x_n) \to (f - x^*)(x_0)$ and $x^* \in \partial f(x_0)$.

(ii) *nearly strongly convex at* x_0, if $\partial f(x_0) \neq \emptyset$ and (x_n) has a convergent subsequence whenever $(f - x^*)(x_n) \to (f - x^*)(x_0)$ and $x^* \in \partial f(x_0)$.

(iii) *U-convex at* x_0 if for every sequence (x_n) in $\operatorname{dom} f$ satisfying $\frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f(\frac{1}{2}x_n + \frac{1}{2}x_0) \to 0$ there exists $x^* \in \partial f(x_0)$ such that $(f - x^*)(x_n) \to (f - x^*)(x_0)$.

(iv) *nearly U-convex at* x_0 if for every sequence (x_n) in $\operatorname{dom} f$ satisfying $\frac{1}{2}f(x_n) + \frac{1}{2}f(x_0) - f(\frac{1}{2}x_n + \frac{1}{2}x_0) \to 0$, there exists $x^* \in \partial f(x_0)$ and a subsequence (x_{n_k}) such that $(f - x^*)(x_{n_k}) \to (f - x^*)(x_0)$.

Proposition H

Let $x_0 \in \operatorname{dom} f$ and $\partial f(x_0) \neq \emptyset$. Consider the following statements.

(i) f is nearly strongly convex at x_0.

(ii) $\alpha(L(x^*, f, x_0, \frac{1}{n})) \to 0$ for every $x^* \in \partial f(x_0)$.

(iii) $\alpha(\partial^X_{\frac{1}{n}} f^*(x^*)) \to 0$ for every $x^* \in \partial f(x_0)$.

(iv)$\alpha(\partial_{\frac{1}{n}} f^*(x^*)) \to 0$ for every $x^* \in \partial f(x_0)$.

(v) $\partial^X f^*(x^*)$ is compact and f^* is strongly sub-differentiable at every $x^* \in \partial f(x_0)$.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). If $\partial f(x_0) \subset \operatorname{int}(\operatorname{dom} f^*)$ then (iv) $\Leftrightarrow$ (v).

Thank you for your attention!