Optimality conditions in optimization under uncertainty

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In honor of Jean-Paul Penot

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5. Conclusions, further research

Part 3: Klamroth, K., Köbis, E., Schöbel, A., Tammer, Chr. (2013): A unified approach for different concepts of robustness and stochastic programming via nonlinear scalarizing functionals. Optimization, 62 (5), 649-671.
Klamroth, K., Köbis, E., Schöbel, A., Tammer, Chr. (2017): A unified approach to uncertain optimization. European J. Oper. Res., 260(2), 403-420.

1. Introduction and previous work

Most optimization problems involve uncertain data due to measurement errors, unknown future developments and modeling approximations.

In risk theory, assets are naturally affected by uncertainty due to market changes, changing preferences of customers and unforeseeable events.

Consequently, it is highly important to introduce uncertain parameters to optimization problems.

(Birge, Louveaux (1997), Ben-Tal, Ghaoui, Nemirovski (2009), Rockafellar, Royset (2013, 2015).)

In KKST (2013, 2017): Different approaches to uncertain optimization can be put in a unifying context using vector optimization, set optimization and nonlinear scalarizing functionals, assuming that the uncertainty set consists of finitely / infinitely many elements.

Our goal: Employing unifying concepts for both stochastic and robust optimization for deriving optimality conditions.

Express robust and stochastic optimization problems by using the mentioned nonlinear scalarizing functional.
Connections to vector optimization problems in general spaces as well as set-valued optimization (J.-P. Penot (2016): Analysis. *From Concepts to Applications*. Springer, Chapter 1: Sets, Orders, Relations and Measures).

2. Scalar optimization problems under uncertainty

Consider an optimization problem $(Q(\xi))$ which depends on uncertain parameters ξ that belong to a given uncertainty set $\mathcal{U}$:

$$\begin{aligned} & f(x,\xi) \to \inf \\ & \text{s.t. } F_i(x,\xi) \le 0, \ \ i = 1,\dots,m, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{$Q(\xi)$}$$

where $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $i = 1,\dots,m$.

$\mathcal{X}(\xi) = \{x \in \mathbb{R}^n \mid F_i(x,\xi) \le 0, \ \ i = 1,\dots,m\}$. $\mathcal{U} \neq \emptyset$ is a not necessarily finite set, $\mathcal{X}(\xi) \neq \emptyset$ for all $\xi \in \mathcal{U}$.

An uncertain optimization problem $P(\mathcal{U})$ is defined as a family of parametrized optimization problems

$$(Q(\xi), \xi \in \mathcal{U}). \tag{1}$$

Two approaches regarding uncertain optimization problems:

- Stochastic Optimization: This idea goes back to Dantzig (1955). Stochastic optimization assumes that the uncertain parameter is probabilistic. Usually, one optimizes some cost function using the expected value of the uncertain parameter (cf. Birge, Louveaux (1997)).

- Robust Optimization: Robustness, pursues a distinctively different approach to optimization problems with uncertainties not relying on a probability distribution but only using the uncertainty set. Typically, one wishes to optimize the worst-case scenario (strict robustness: Ben-Tal, Ghaoui, Nemirovski (2009)).

Example 1 (Uncertain linear optimization problem). Consider $f(x,\xi) := c(\xi)^T x$, $x \in \mathbb{R}^n$, $c(\xi) = c^0 + \sum_{i=1}^L \xi_i c^i$, $c^i \in \mathbb{R}^n$, $i = 0, \ldots, L$. $\mathcal{U} := \{\xi \in \mathbb{R}^L | -1 \leq \xi_i \leq 1,\ i = 1, \ldots, L\}$. To solve

$$(c^0 + \sum_{i=1}^{L} \xi_i c^i)^T x \to \inf_{x \in \mathbb{R}^n} . \qquad (Q(\xi))$$

Strictly robust solution. Minimize the worst-case objective function, i.e.,

$$\sup_{\xi \in \mathcal{U}} (c^0 + \sum_{i=1}^{L} \xi_i c^i)^T x \to \inf_{x \in \mathbb{R}^n} .$$

Minimizing the expectation. In the case a probability distribution over $\mathcal{U}$ is known: Minimize the expected objective value, i.e.,

$$\mathbb{E}(c^0 + \sum_{i=1}^{L} \xi_i c^i)^T x \to \inf_{x \in \mathbb{R}^n} .$$

3. Three unifying concepts for uncertain optimization
3.1 Vector optimization as unifying concept

Idea: To generalize the approach for finite uncertainty sets: Let $\mathcal{U} = \{\xi_1, \ldots, \xi_L\}$. Then, for each scenario, we can introduce an objective function. For some point x, we then obtain a vector $F_x \in \mathbb{R}^L$ which contains $f(x, \xi_i)$ in its ith coordinate.
KKST (2013): Robust solutions can be characterized in terms of multiobjective optimization for many robustness concepts.

KKST (2017): $\mathcal{U}$ is not a finite set, we obtain not vectors F_x but functions, i.e., $F_x : \mathcal{U} \to \mathbb{R}$ where $F_x(\xi) := f(x, \xi)$ contains the objective value of x in scenario ξ.
To compare two points x and y: Order relations in the real linear functional space $\mathbb{R}^{\mathcal{U}}$ of all mappings $F : \mathcal{U} \to \mathbb{R}$.

Let $Y = \mathbb{R}^{\mathcal{U}}$ be the space of all functions $F : \mathcal{U} \to \mathbb{R}$.
For some fixed $x \in \mathbb{R}^n$:

$$F_x \in Y : \quad F_x(\xi) := f(x, \xi).$$

To compare elements of Y: Consider different orderings on the space Y denoted by α. Let C be a proper pointed closed convex cone. Such a cone C induces partial ordering $\alpha :=\leq_C$ by

$$y_1 \in y_2 - C \iff y_1 \leq_C y_2.$$

Example 2: The natural order relation on Y is induced by

$$C_Y := \{F \in Y \mid \forall \xi \in \mathcal{U} : \; F(\xi) \geq 0\} :$$

$$\forall\, F, G \in Y : \; F \,\alpha\, G \; :\iff \; G \in F + C_Y \iff F(\xi) \leq G(\xi) \text{ for all } \xi \in \mathcal{U}.$$

Definition 1 Let $\mathcal{F}$ be a nonempty subset of Y. An element $F \in \mathcal{F}$ is a minimal element of $\mathcal{F}$ in Y w.r.t. α if

$$\text{for } G \in \mathcal{F}: \; G \;\alpha\; F \implies F \;\alpha\; G.$$

If α is induced by a proper cone C in Y with $\operatorname{int} C \neq \emptyset$, an element $F \in \mathcal{F}$ is a weakly minimal element of $\mathcal{F}$ in Y w.r.t. α if

$$(F - \operatorname{int} C) \cap \mathcal{F} = \emptyset.$$

If α is induced by a cone, then an element $F \in \mathcal{F}$ is a minimal element of $\mathcal{F}$ in Y w.r.t. α if and only if $(F - C) \cap \mathcal{F} \subseteq F + C$.

Remark: Rockafellar, Royset (2013, 2015): Unifying framework for handling uncertainty in a decision making process. $f(x, \cdot)$ is considered as random variable and by means of risk measures, different models would be possible that address the issue how to treat that random variable. Since random variables are also functions, the connection to the vector approach is evident.

3.2 Set-based optimization as unifying concept

We are interested in all possible objective values which can appear if solution x is chosen. These are given by

$$B_x := f(x,\mathcal{U}) := \{f(x,\xi) \mid \xi \in \mathcal{U}\}.$$

To compare two solutions x and y in this setting, define order relations between their corresponding sets B_x and B_y.

Power set of $\mathbb{R}$ without the empty set: $Z := \{A \subseteq \mathbb{R} \mid A \text{ is nonempty } \}$. For a given $x \in \mathbb{R}^n$, we have

$$B_x \in Z: \quad B_x = \text{img}(F_x)$$

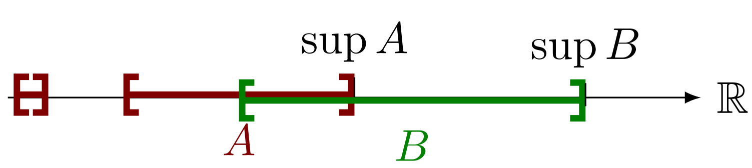
(the image of the mapping F_x under $\mathcal{U}$). $B_x \subseteq \mathbb{R}$ is an interval in case that $f(x,\cdot)$ is a continuous function.

In order to compare elements of Z, we consider certain set less relations denoted by β.

Example 3: (Upper-type set-relation (Kuroiwa 1999, 1998)) Let $A, B \in Z$ be arbitrarily chosen nonempty closed sets. Then the u-type set-relation $\beta :=\preceq^u$ is defined by

$$A \; \beta \; B :\Longleftrightarrow A \subseteq B - \mathbb{R}_+ \Longleftrightarrow \forall a \in A \; \exists b \in B : a \leq b$$

which is equivalent to $\sup A \leq \sup B$. Note that β is induced by the cone $\mathbb{R}_+$ in Z.

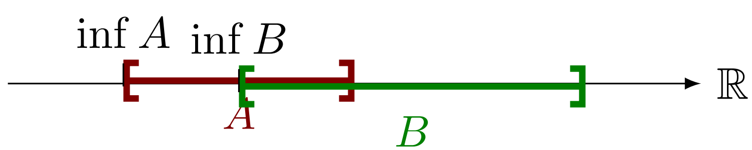


Visualization of $A \; \beta \; B$ *with* $\beta =\preceq^u$.

Example 4:(Lower-type set-relation (Kuroiwa 1999, 1998)) Let $A, B \in Z$ be arbitrarily chosen nonempty closed sets. Then the l-type set-relation $\beta := \preceq^l$ is defined by

$$A \; \beta \; B :\Longleftrightarrow B \subseteq A + \mathbb{R}_+ \Longleftrightarrow \forall b \in B \; \exists a \in A : a \leq b$$

which is equivalent to $\inf A \leq \inf B$.



Visualization of $A \; \beta \; B$ with $\beta = \preceq^l$.

Let $\mathcal{B}$ be a nonempty subset of Z.

Definition 2: $A \in \mathcal{B}$ is a minimal element in $\mathcal{B}$ w.r.t. β if

$$\text{for } B \in \mathcal{B} : \;\; B \; \beta \; A \;\; \Longrightarrow \;\; A \; \beta \; B.$$

3.3 A nonlinear scalarizing functional as unifying concept

Let Y be a linear topological space, $k \in Y \setminus \{0\}$ and let $\mathcal{F}, B$ be proper subsets of Y, B closed. We assume that

$$B + [0, +\infty) \cdot k \subseteq B. \tag{2}$$

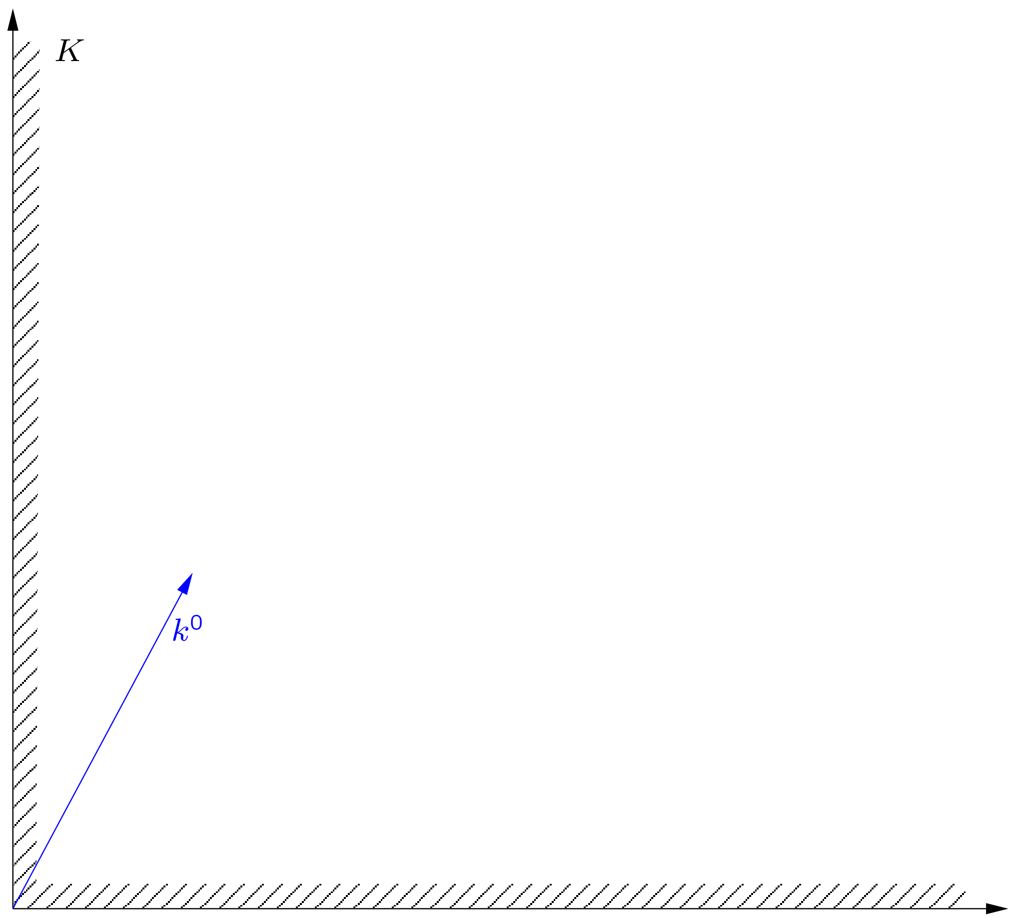
We consider the functional $z^{B,k} : Y \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} =: \bar{\mathbb{R}}$

$$z^{B,k}(y) := \inf\{t \in \mathbb{R} | y \in tk - B\}. \tag{3}$$

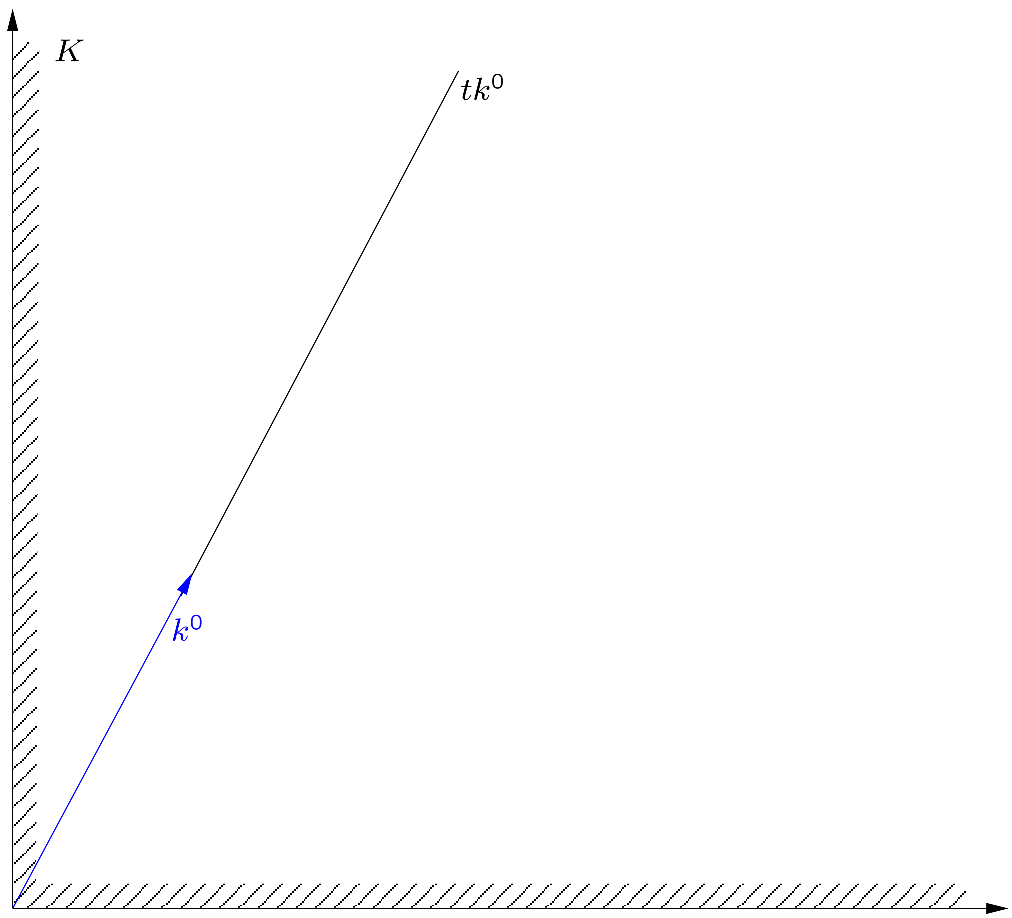
This functional was used as separating functional by Gerstewitz (1983,1984), Pascoletti, Serafini (1984), Göpfert, Riahi, Tammer, Zălinescu (2003), Penot, Sterna-Karwat (1986, 1989), Sterna-Karwat (1987), Krasnosel'ski (1964), Rubinov (1977).

Definition 3: An element $F \in \mathcal{F}$ is a minimal element of $\mathcal{F}$ in Y w.r.t. (3) if F solves the problem

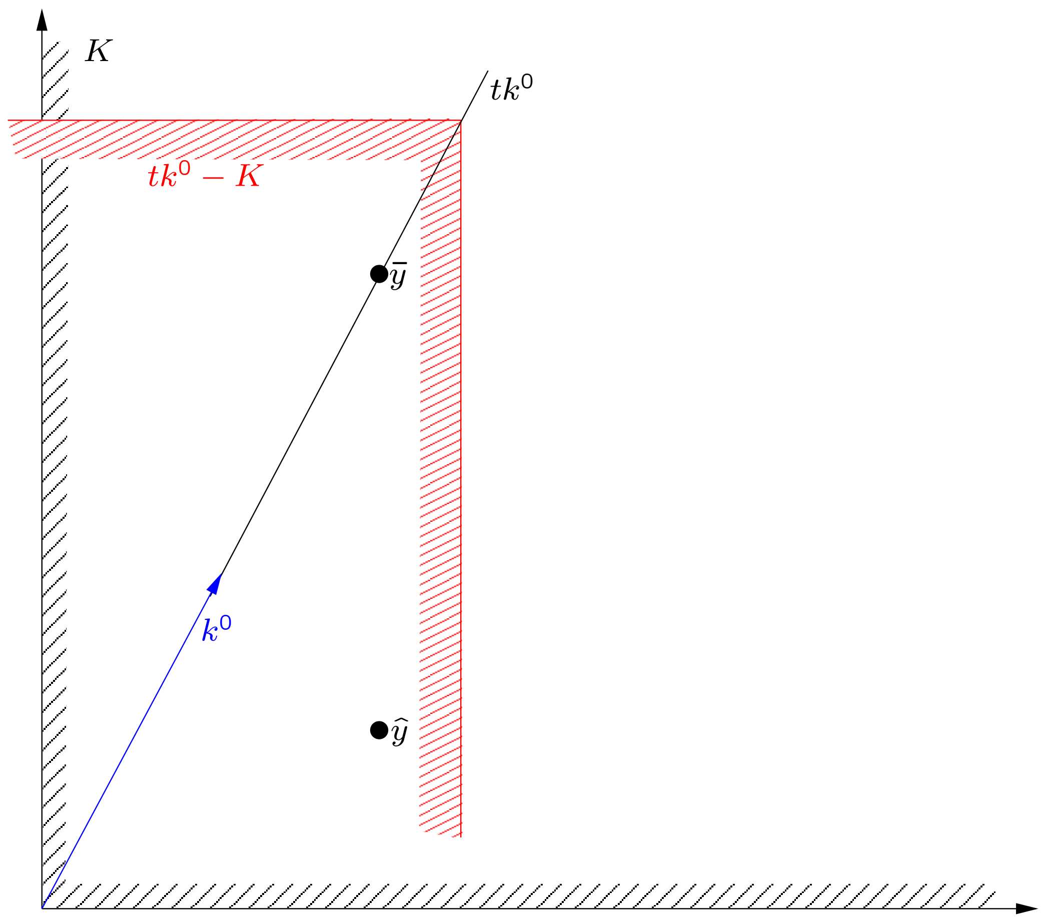
$$z^{B,k}(y) \to \inf_{y \in \mathcal{F}} . \qquad (P_{k,B,\mathcal{F}})$$



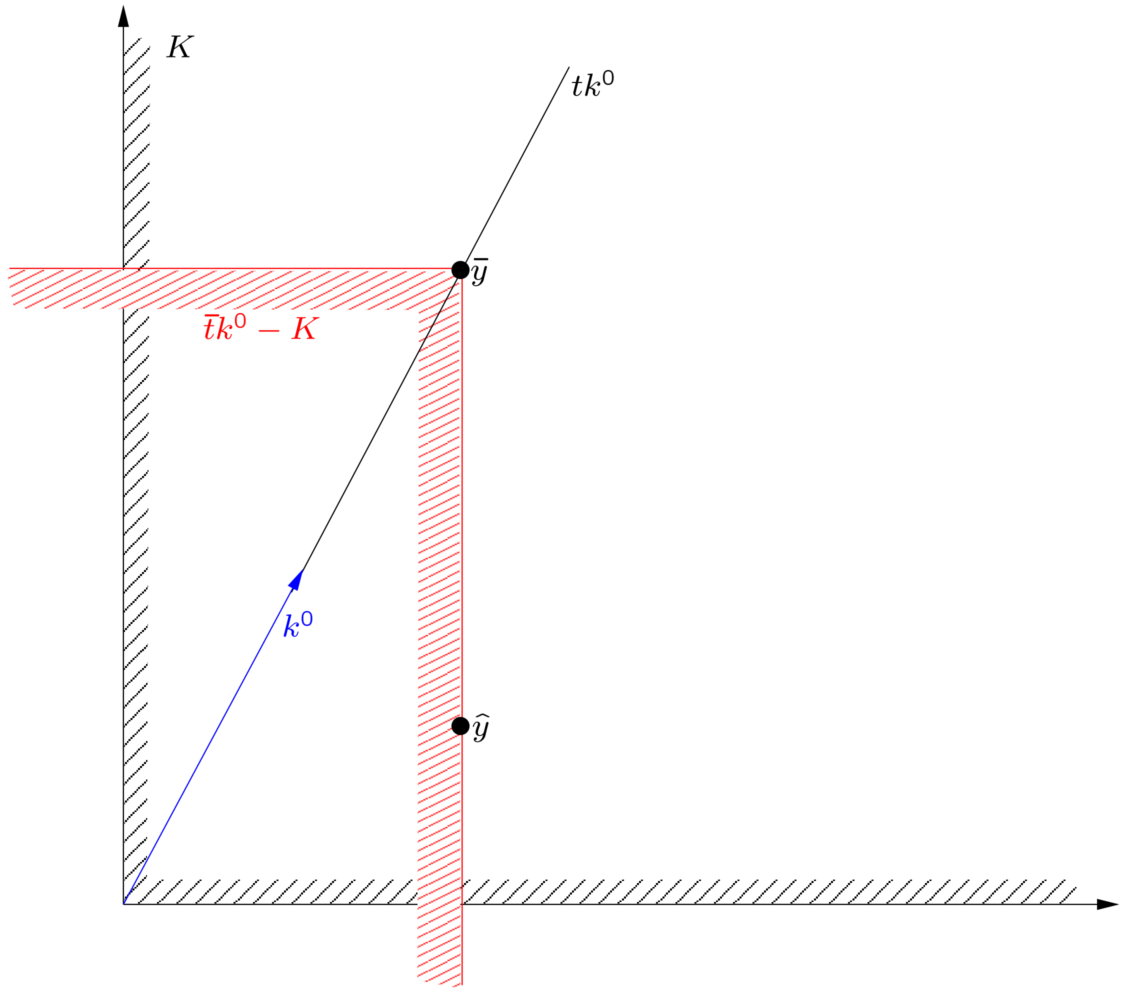
$$Y = \mathbb{R}^2,\ B = K,\ z^{B,k} : Y \to \overline{\mathbb{R}},\ z^{B,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - B\}$$



$$z^{B,k} : Y \to \overline{\mathbb{R}}, \qquad z^{B,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - B\}$$



$$z^{B,k} : Y \to \overline{\mathbb{R}}, \qquad z^{B,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - B\}$$



$$z^{B,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - B\}, \quad z^{B,k}(\bar{y}) = z^{B,k}(\hat{y}) = \bar{t}.$$

Operator Theory: Krasnosel'ski (1964), Rubinov (1977).
Separation theorems, vector optimization: Gerstewitz (1983, 1984), Pascoletti, Serafini (1984), Gerstewitz / Iwanow (1985), Göpfert, Ta., Zălinescu (1999).
In Economics: Luenberger (1992): Shortage function associated to the production possibility set $\mathcal{Y} \subset \mathbb{R}^m$ and $g \in \mathbb{R}^m_+ \setminus \{0\}$:

$$\sigma(g; y) := \inf\{\xi \in \mathbb{R} \mid y - \xi g \in \mathcal{Y}\},$$

Luenberger (1992): Benefit function.
Mathematical Finance: Coherent risk measures associated to the set of random variables corresponding to acceptable investments. Artzner et. al. (1999), Marohn (2022).
Functional Analysis: Rubinov, Singer (2001) *topical functionals*.

3.4 Unifying concepts for strict robustness

Strict robustness: Soyster (1973), Ben-Tal, Nemirovski (1998), El Ghaoui (1997).

Idea: The worst possible objective function value is minimized in order to get a solution that is "good enough" even in the worst-case scenario. Strictly robust counterpart of $(Q(\xi), \xi \in \mathcal{U})$:

$$\begin{aligned} \rho_{RC}(x) = \sup_{\xi \in \mathcal{U}} f(x,\xi) &\to \inf \\ \text{s.t.} \forall \xi \in \mathcal{U}: \; F_i(x,\xi) &\le 0, \; i = 1, \ldots, m, \\ x &\in \mathbb{R}^n. \end{aligned} \tag{RC}$$

We call a feasible solution of (RC) strictly robust. Set of strictly robust solutions:

$$\mathfrak{A}_1 := \{x \in \mathbb{R}^n \mid \forall \xi \in \mathcal{U}: \; F_i(x,\xi) \le 0, \; i = 1, \ldots, m\}. \tag{4}$$

Vector optimization as unifying concept

The strictly robust counterpart problem (RC) can be formulated as a vector optimization problem in the infinite dimensional space $Y = \mathbb{R}^{\mathcal{U}}$ as follows:
For $F_x : \mathcal{U} \to \mathbb{R}$, $F_x(\xi) = f(x, \xi)$, consider the set of strictly robust outcome functions in Y: $\mathcal{F}_1 := \{F_x \in Y \mid x \in \mathfrak{A}_1\}$, F_x, $F_y \in Y$ and the sup-order relation $\alpha_1 :=\leq_{sup}$ on Y:

$$F_x \; \alpha_1 \; F_y \; :\Longleftrightarrow \sup_{\xi \in \mathcal{U}} F_x(\xi) \leq \sup_{\xi \in \mathcal{U}} F_y(\xi).$$

Minimal elements of $\mathcal{F}_1$ w.r.t. $\leq_{sup}$ (see Definition 1).

Compute minimal elements of $\mathcal{F}_1$ w.r.t. $\leq_{sup}$. $\quad$ ($\leq_{sup}$-VOP)

Lemma 1 *Assume that every* $F_x \in \mathcal{F}_1$ *attains its supremum. If* $F_x \in \mathcal{F}_1$ *is a minimal element of* $\mathcal{F}_1$ *in* Y *w.r.t.* α_1*, then* F_x *is a weakly minimal element of* $\mathcal{F}_1$ *in* Y *w.r.t. the natural ordering* α *of* Y *induced by the ordering cone* C_Y*.*

Theorem 2 *A strictly robust solution* $x \in \mathfrak{A}_1$ *is an optimal solution of (RC) if and only if the corresponding strictly robust outcome function* $F_x \in \mathcal{F}_1$ *is a minimal element of* $\mathcal{F}_1$ *w.r.t. the sup-order relation* α_1*.*

Remark. In the light of Lemma 1, for each optimal solution x of the strictly robust counterpart problem (RC), F_x is a weakly minimal element of $\mathcal{F}_1$ w.r.t. the natural ordering α in Y.

Set-valued optimization as unifying concept

Interpretation of strictly robust counterpart problem (RC) as a set-valued optimization problem (see Definition 2).
For closed sets B_x, $B_y \in Z$, the upper-type set-relation $\beta_1 :=\preceq^u$ is given by

$$B_x \; \beta_1 \; B_y :\Longleftrightarrow B_x \subseteq B_y - \mathbb{R}_+ \Longleftrightarrow \sup B_x \leq \sup B_y.$$

Set of strictly robust outcome sets: $\mathcal{B}_1 := \{B_x \in Z \mid x \in \mathfrak{A}_1\}$.

$$\text{Compute minimal elements of } \mathcal{B}_1 \text{ w.r.t. } \preceq^u. \qquad (\preceq^u\text{-SP})$$

Theorem 3 *Suppose that the sets B_x are closed for all $x \in \mathfrak{A}_1$. A strictly robust solution $x \in \mathfrak{A}_1$ is an optimal solution of (RC) if and only if the corresponding strictly robust outcome set $B_x \in \mathcal{B}_1$ is a minimal element of $\mathcal{B}_1$ with respect to the order relation β_1.*

Nonlinear scalarization as unifying concept

Interpretation of the strictly robust counterpart problem (RC) using the nonlinear scalarizing functional (3)

$$z^{B,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - B\},$$

where $k \in Y \setminus \{0\}$, $B \subset Y$ proper and closed, $B + [0, +\infty) \cdot k \subseteq B$.

Employing the scalarizing functional (3), we consider the following scalar minimization problem (see Definition 3)

$$z^{B,k}(F) \to \inf_{F \in \mathcal{F}_1} . \qquad (P_{k,B,\mathcal{F}_1})$$

Let $F_x : \mathcal{U} \to \mathbb{R}$, $F_x(\xi) = f(x, \xi)$.

Theorem 4 *Let* $B_1 := C_Y$, $k_1 :\equiv 1$ *and* $\mathcal{F}_1 := \{F_x \in Y |\ x \in \mathfrak{A}_1\}$. *Then* x *is an optimal solution of the strictly robust counterpart problem (RC) if and only if* F_x *solves problem* $(P_{k_1,B_1,\mathcal{F}_1})$.

Proof: $B_1 + [0, +\infty) \cdot k_1 \subseteq B_1$ holds, thus inclusion (2) is satisfied. Furthermore, we have

$$
\begin{aligned}
z^{B_1,k_1}(F_x) &= \inf\{t \in \mathbb{R} | F_x \in tk_1 - B_1\} \\
&= \inf\{t \in \mathbb{R} | F_x \in tk_1 - C_Y\} \\
&= \inf\{t \in \mathbb{R} | F_x - tk_1 \in -C_Y\} \\
&= \inf\{t \in \mathbb{R} | \forall \xi \in \mathcal{U} :\ \ F_x(\xi) \leq t\} \\
&= \inf\{t \in \mathbb{R} | \forall \xi \in \mathcal{U} :\ \ f(x, \xi) \leq t\} \\
&= \sup_{\xi \in \mathcal{U}} f(x, \xi).
\end{aligned}
$$

Thus, F_x is minimal for $(P_{k_1,B_1,\mathcal{F}_1})$ if and only if $x \in \mathfrak{A}_1$ minimizes $\sup_{\xi \in \mathcal{U}} f(x, \xi)$, i.e., x is an optimal solution to (RC).

ϵ-constraint robustness based on the nonlinear scalarization

Let Y be the space of all functions $F: \mathcal{U} \to \mathbb{R}$, $C_Y := \{F \in Y | \forall \xi \in \mathcal{U}: F(\xi) \geq 0\}$. Fix $\bar{\xi} \in \mathcal{U}$. Consider $\epsilon: \mathcal{U} \to \mathbb{R}$, let $\mathcal{F}_2 := \{F_x \in Y | x \in \mathfrak{A}_1\}$, with $\mathfrak{A}_1 \subseteq \mathbb{R}^n$, $F_x(\xi) := f(x, \xi)$. Furthermore, let $B_2 := \{y \in Y | y \in C_Y - \epsilon\}$ and $k_2: \mathcal{U} \to \mathbb{R}$,

$$k_2 = \begin{cases} 1 & \text{for } \xi = \bar{\xi}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Theorem 5 *Let $\epsilon: \mathcal{U} \to \mathbb{R}$. Then for $k = k_2$, $B = B_2$,* (2) *holds and with $\mathcal{F} = \mathcal{F}_2$, problem $(P_{k,B,\mathcal{F}})$ is equivalent to*

$$\begin{aligned} &\inf f(x, \bar{\xi}) - \epsilon(\bar{\xi}) \\ &\text{s.t. } \forall\, \xi \in \mathcal{U}: \ F_i(x, \xi) \leq 0, \ i = 1, \ldots, m, \ x \in \mathbb{R}^n, \\ &\qquad \forall\, \xi \in \mathcal{U} \setminus \{\bar{\xi}\}: \ f(x, \xi) \leq \epsilon(\xi). \end{aligned} \tag{ϵRC}$$

4. Optimality conditions for solutions of robust counterpart problems

Generic approach to subdifferentials (see e.g. Dolecki, Malivert (1993), Durea, Tammer (2009)):

Let $\mathcal{X}$ be a class of Banach spaces which contains the class of finite dimensional normed vector spaces.

Abstract subdifferential ∂: A map which associates to every lower semicontinuous (lsc) function $h : X \in \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ and to every $x \in X$ a (possible empty) subset $\partial h(x) \subset X^*$. We use the notation $\operatorname{Dom} h := \{x \in X \mid h(x) \neq +\infty\}$. Let $X, Y \in \mathcal{X}$ and denote by $\mathcal{F}(X, Y)$ a class of functions acting between X and Y having the property that by composition at left with a lsc function from Y to $\overline{\mathbb{R}}$ the resulting function is still lsc.

(H1) If h is convex, then $\partial h(x)$ coincides with the Fenchel subdifferential.

(H2) If x is a local minimum point for h, then $0 \in \partial h(x)$; $\partial h(u) = \emptyset$ if $u \notin \mathrm{Dom}\, h$.

(H3) If $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ is convex and $f \in \mathcal{F}(X, Y)$, then

$$\forall x : \quad \partial(\varphi \circ f)(x) \subseteq \bigcup_{y^* \in \partial\varphi(f(x))} \partial(y^* \circ f)(x).$$

(H4) If $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ is convex, $f \in \mathcal{F}(X, Y)$, and $\Omega \subset X$ is a closed set containing x, then

$$\partial(\varphi \circ f + I_\Omega)(x) \subseteq \partial(\varphi \circ f)(x) + \partial I_\Omega(x).$$

(H5) If h is lower semicontinuous and $g : X \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz, then for every $x \in \mathrm{Dom}\, h \cap \mathrm{Dom}\, g$,

$$\partial(h + g)(x) \subseteq \partial h(x) + \partial g(x).$$

For a closed set $\Omega \subset X$, the set $\partial I_\Omega(x)$ is denoted by $N_\partial(x;\Omega)$ and is called the set of normal directions to Ω at $x \in \Omega$ with respect to ∂.

The properties (H3), (H4) and (H5) are called "exact calculus rules" for sums and for composition. Jean-Paul Penot: Subdifferential calculus without qualification assumptions. Journal of Convex Analysis Volume 3 (1996), No. 2, 207-219.

- the limiting subdifferential (Kruger, Mordukhovich) when $\mathcal{X}$ is the class of Asplund spaces, Y is finite dimensional and $\mathcal{F}(X,Y)$ is the class of Lipschitz functions from X into Y;

- the approximate subdifferential (Ioffe) when $\mathcal{X}$ is the class of Banach spaces and $\mathcal{F}(X,Y)$ is the class of strongly compactly Lipschitz functions from X into Y.

Theorem 6 *Let* $B \subset Y$ *be a closed convex proper set and* $k \in Y \setminus \{0\}$ *s.t. (2) holds. Consider the functional* $z^{B,k}$ *in (3) and let* $\overline{y} \in \operatorname{Dom} z^{B,k}$*. Then*

$$\partial z^{B,k}(\overline{y}) = \{y^* \in Y^* \mid y^*(k) = 1, \forall b \in B : y^*(b) + y^*(\overline{y}) - z^{B,k}(\overline{y}) \geq 0\}.$$

Corollary 7 *Let* $C \subset Y$ *be a closed convex cone with nonempty interior. Then, for every* $k \in \operatorname{int} C$ *the functional* $z^{C,k}$ *(see (3) with* $B := C$*) is continuous, sublinear, strictly-*$\operatorname{int} C$*-monotone and for every* $\overline{y} \in Y$, $\partial z^{C,k}(\overline{y})$ *is nonempty and*

$$\partial z^{C,k}(\overline{y}) = \{y^* \in C^+ \mid y^*(k) = 1, y^*(\overline{y}) = z^{C,k}(\overline{y})\}.$$

In particular, $\partial z^{C,k}(0) = \{y^* \in C^+ \mid y^*(k) = 1\}$.

Necessary conditions for solutions of the strictly robust counterpart problem based on nonlinear scalarization functionals:

Theorem 8 *Let Y be the space of all functions $F: \mathcal{U} \to \mathbb{R}$, $B_1 := C_Y$, $k_1 :\equiv 1$ and $\mathcal{F}_1 := \{F_x \in Y |\ x \in \mathfrak{A}_1\}$. Consider an optimal solution x to the strictly robust counterpart problem (RC) and the abstract subdifferential ∂ such that (H1), (H2), (H3), (H4) and (H5) are fulfilled.*
Then, there exists an element $y^* \in (C_Y)^+$ with $y^*(k_1) = 1$ and $y^*(F_x) = z^{B_1,k_1}(F_x)$ such that

$$-y^* \in N(F_x; \mathcal{F}_1).$$

.

Remark In the case of a finite number of scenarios (L scenarios), we have $y^* \in \mathbb{R}^L_+$ in the conditions of Theorem 8.

Theorem 9 *Let* Y *be the space of all functions* $F: \mathcal{U} \to \mathbb{R}$, $\epsilon: \mathcal{U} \to \mathbb{R}$, $\mathcal{F}_2 := \{F_x \in Y |\ x \in \mathfrak{A}_1\}$, *with* $\mathfrak{A}_1 \subseteq \mathbb{R}^n$, $F_x(\xi) := f(x, \xi)$. *Furthermore, let* $k_2: \mathcal{U} \to \mathbb{R}$,

$$k_2 := \begin{cases} 1 & \text{for } \xi = \bar{\xi}, \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

and $C_Y := \{F \in Y | \forall \xi \in \mathcal{U}: \ F(\xi) \geq 0\}$. *Moreover, we define* $B_2 := \{y \in Y |\ y \in C_Y - \epsilon\}$.

Consider an optimal solution x *to the* ϵ*-constraint robust counterpart problem (*ϵ*RC) and the abstract subdifferential* ∂ *such that (H1), (H2), (H3), (H4) and (H5) are fulfilled. Then, there exists an element* $y^* \in Y^*$ *with* $y^*(k_2) = 1$ *and* $y^*(F_x) + y^*(b) \geq z^{B_2, k_2}(F_x)$ *for every* $b \in B_2$ *such that*

$$-y^* \in N(F_x; \mathcal{F}_2).$$

5. Conclusions, further research

Unifying approaches for regret robustness, reliability, adjustable robustness, minimizing the expectation, stochastic dominance.

Using the approach based on vector optimization, a necessary condition for optimal solutions $x \in \mathfrak{A}_1$ of (RC) can be formulated by a vector variational inequality with a mapping $W : X \to L(X, Y)$:

Find $x \in \mathfrak{A}_1$ such that $(W(x))(u - x) \notin \operatorname{int} C_Y$ for every $u \in \mathfrak{A}_1$.

Algorithms (projection methods), see Hebestreit (2020).

To study dependence of solutions to the vector problem on parameters (continuity properties) employing results by Penot, Sterna-Karwat (1986, 1989) to robust counterpart problems based on nonlinear scalarization.

Necessary conditions for solutions of the robust counterpart problems (RC) based on the upper set relation using Bao, Tammer (2019), Theorem 4.1.

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Happy birthday, dear Jean-Paul!