Weak convexity and approximate subdifferentials

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JPP-Fest *Challenges and advances in modern variational analysis*
March 15th, 2023

A quotation from 1997

Hunting for a Smaller Convex Subdifferential

V. F. Demyanov & V. Jeyakumar

Journal of Global Optimization **10**, 305–326 (1997)

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today we join the expedition, now hunting for ε-subdifferentials

Why bother about ε-subgradients?

Two prolific hunters in the convex case

JBHU (calculus!) and CLL (algorithms)

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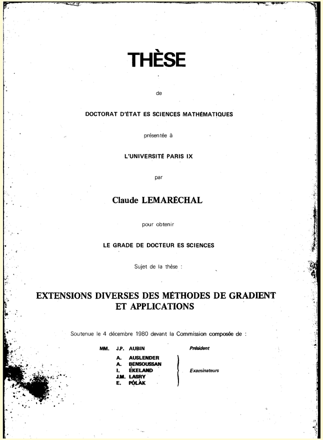
JBHU (calculus!) and CLL (algorithms)Chapter 5, Dec 4th, 1980:



THÈSE

de

DOCTORAT D'ÉTAT ES SCIENCES MATHÉMATIQUES

présentée à

L'UNIVERSITÉ PARIS IX

par

Claude LEMARÉCHAL

pour obtenir

LE GRADE DE DOCTEUR ES SCIENCES

Sujet de la thèse :

EXTENSIONS DIVERSES DES MÉTHODES DE GRADIENT
ET APPLICATIONS

Soutenue le 4 décembre 1980 devant la Commission composée de :

MM.	J.P. AUBIN	*Président*
	A. AUSLENDER	*Examinateurs*
	A. BENSOUSSAN	
	I. EKELAND	
	J.M. LASRY	
	E. POLAK	

1.2. *Construction des ε-sous-différentiels*.

Etant donnée une fonction convexe, on peut en général calculer en un point x : la valeur de la fonction, et un sous-gradient ; plus rarement, on peut calculer tout $\partial f(x)$. Il est exceptionnel de pouvoir calculer directement tout $\partial_\varepsilon f(x)$ pour un $\varepsilon > 0$ (cf. [11]). La question se pose donc de savoir comment calculer des éléments de $\partial_\varepsilon f(x)$ qui ne soient pas dans $\partial f(x)$. Ce paragraphe montre qu'on peut le faire en calculant des éléments de $\partial f(y)$, pour des points y bien choisis.

Théorème 1.2.1.

Soient x et y appartenant à dom f, $g \in \partial f(y)$. Une condition nécessaire et suffisante pour que g soit également dans $\partial_\varepsilon f(x)$ est :

(11) $$f(y) \geq f(x) + (g, y-x) - \varepsilon$$

Démonstration : La condition et évidemment nécessaire : la relation de définition (7) doit au moins être satisfaite pour z=y.

<u>Remarque 1.2.2.</u> : Ce théorème très simple est fondamental, et nous en ferons un usage constant. Nous l'appellerons <u>théorème de transport des sous-gradients</u>.

Un façon simple d'interpréter la relation (11) est de considérer le nombre

$$\alpha(y,g,x) = f(x) - [f(y) + (g,x-y)]$$

Ce nombre est positif. Il représente l'erreur faite en remplaçant f(x) par la valeur en x de la linéarisation de f en y. Le théorème 1.2.1. s'écrit : soit $g \in \partial f(y)$. Alors :

$$g \in \partial_\varepsilon f(x) \quad \text{si et seulement si} \quad \alpha(y,g,x) \leq \varepsilon$$

ce qui peut s'énoncer : $g \in \partial_\varepsilon f(x)$ si f(x) est approché à ε près par la linéarisation de f en y. //

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The inclusion $0 \in \partial f(\bar{x})$ can **fail** as optimality certificate

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$$\left(\quad x^k, g(x^k) \in \partial \;\; f(x^k) \right) : \begin{cases} x^k \to \bar{x} \\ g(x^k) \to \bar{g}\,. \end{cases} \Longrightarrow \bar{g} \in \partial f(\bar{x})$$

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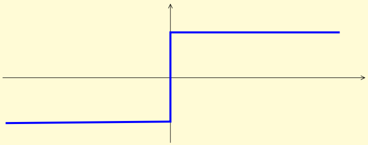

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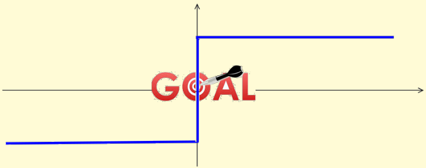

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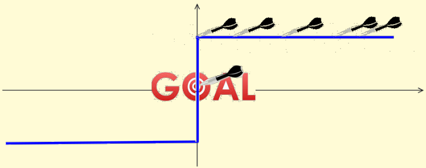

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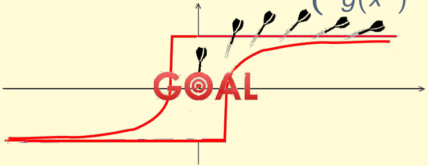

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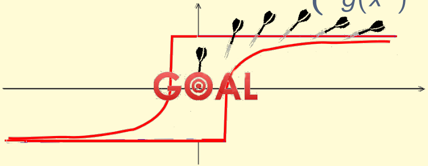
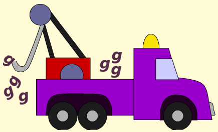


In an algorithmic scheme, ε-subgradients are built using subgradient information

The transportation formula for convex f: from $\partial f(x_0)$ to $\partial_\varepsilon f(u_0)$



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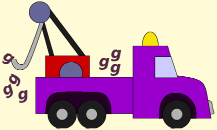

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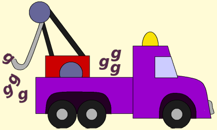

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Algorithmic optimality certificate

- checks if, for $x_\varepsilon^* \in \partial_\varepsilon f(x_\varepsilon)$, $\|x_\varepsilon^*\|$ and ε are small
- for some stepsize t, these objects are driven to 0 by a descent mechanism ensuring

$$0 \leq t\|x_\varepsilon^*\|^2 + \varepsilon \leq \text{fraction of}\Big(f(x^k) - f(x^{k+1})\Big)$$

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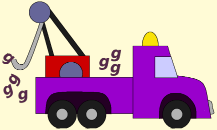

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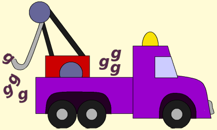

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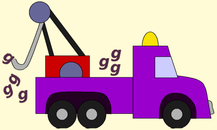

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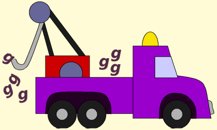

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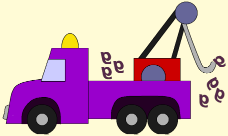

$\Leftarrow$ Bröndsted-Rockafellar theorem

Bröndsted-Rockafellar's-like results

(I) For a closed proper convex function f

- given $u_0^\star \in \partial_\varepsilon f(u_0)$
- there exist $x_\varepsilon \in \mathbb{R}^n$, $x_\varepsilon^\star \in \partial f(x_\varepsilon)$ such that

$$\begin{aligned} \|x_\varepsilon - u_0\| &\leq \sqrt{\varepsilon} \\ \|x_\varepsilon^\star - u_0^\star\| &\leq \sqrt{\varepsilon} \end{aligned}$$

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$$u_0^* - p \in \partial f(u_0 + p)\,, \quad \|p\| \leq \sqrt{\varepsilon}\,.$$

(useful for showing linear convergence rate)

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(III) **a more detailed result:**

J.-P. Penot. "Subdifferential Calculus Without Qualification Assumption"

Journal of Convex Analysis 3 (1996), pp. 207-220

(III) For a closed proper convex function f,

- given $u_0^\star \in \partial_\varepsilon f(u_0)$
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\|x_\varepsilon - u_0\| + \frac{1}{\sqrt{\varepsilon}}|\langle u_0^\star, x_\varepsilon - u_0\rangle| &\leq \sqrt{\varepsilon} \\
\|x_\varepsilon^\star - (1+\gamma)u_0^\star\| &\leq \sqrt{\varepsilon} \\
|\langle x_\varepsilon^\star - u_0^\star, x_\varepsilon - u_0\rangle| &\leq \varepsilon \\
|\langle x_\varepsilon^\star, x_\varepsilon - u_0\rangle| &\leq 2\varepsilon \\
|f(x_\varepsilon) - f(u_0)| &\leq 2\varepsilon
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is there some form of benign nonconvexity that preserves BR's-like results?

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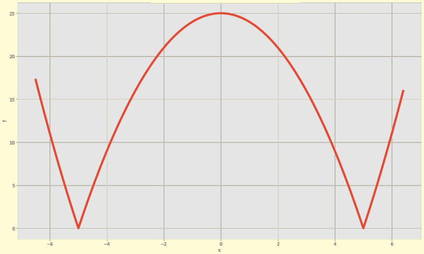

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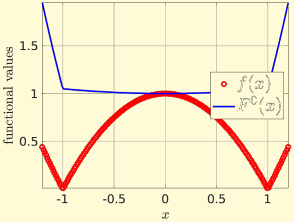

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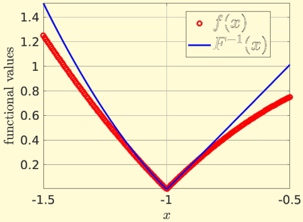

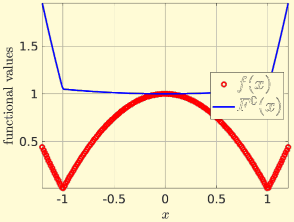

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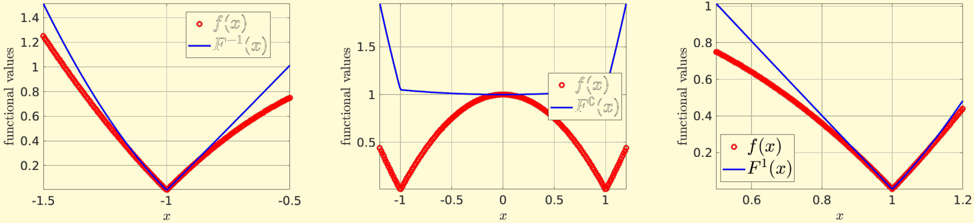

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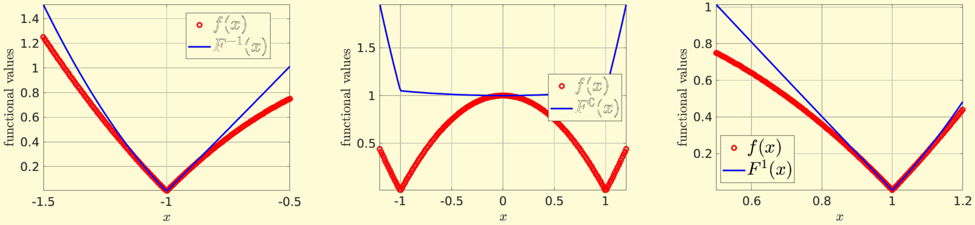

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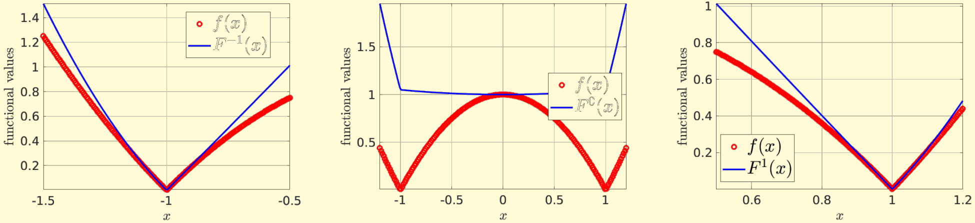

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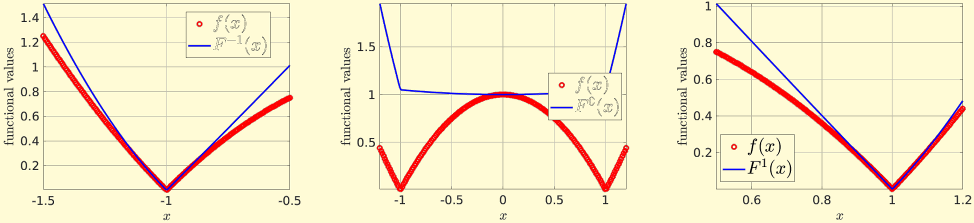

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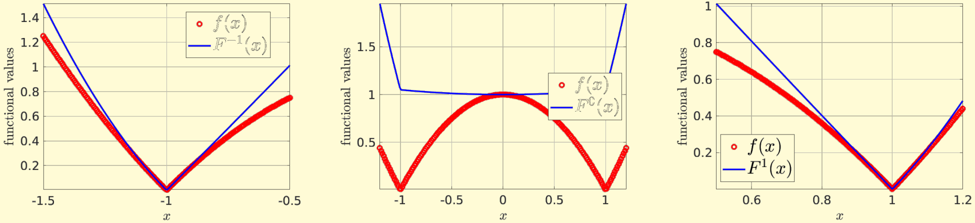

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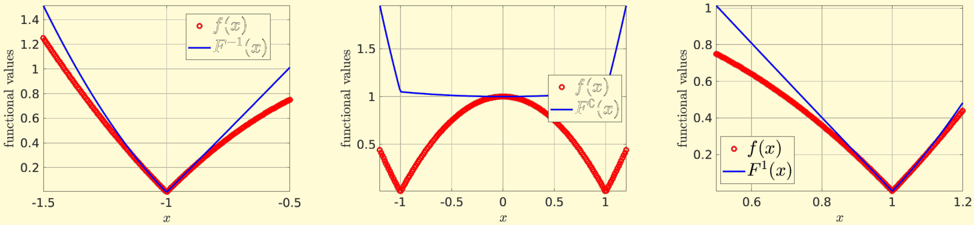

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  - h is a convex function with Lipschitz constant L_h
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In both cases ρ is independent of dimension and m

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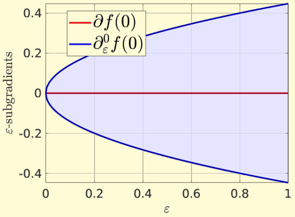

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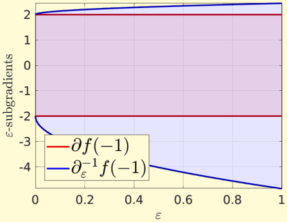

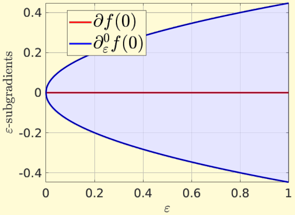

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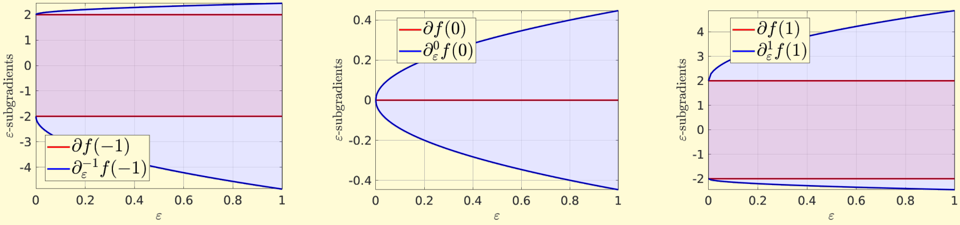

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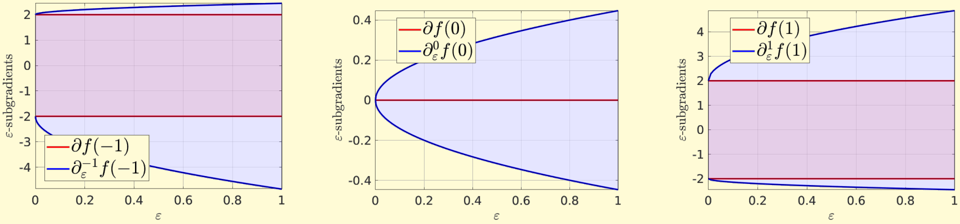

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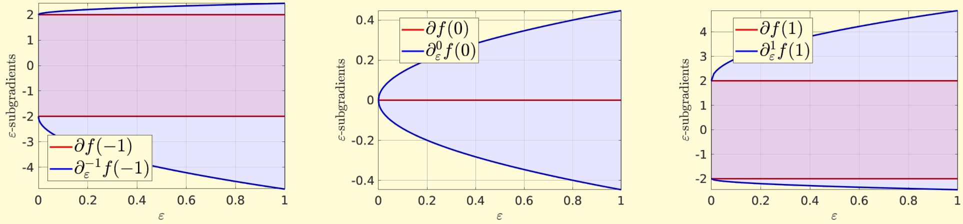

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More results

- **Extension of JPP's Corollary 1.2** For a closed proper **ρ-weakly** convex function f,
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- **Algorithmic optimality certificate** checks if for $x_\varepsilon^* \in \partial_\varepsilon^{x_\varepsilon} f(x_\varepsilon)$, $\|x_\varepsilon^*\|$ and ε are small
  - since $\partial_\varepsilon^{x_\varepsilon} f(x_\varepsilon) = \partial_\varepsilon F^{x_\varepsilon}(x_\varepsilon)$

$$f(y) + \frac{\rho}{2}\|y - x_\varepsilon\|^2 \geq f(x_\varepsilon) + \langle x_\varepsilon^*, y - x_\varepsilon \rangle - \varepsilon$$

(on going work)

ε-subgradient descent schemes for w.c. optimization *à la* S. M. Robinson.

"Linear convergence of epsilon-subgradient descent methods for a class of convex functions"

Mathematical Programming 86.1 (1999)

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The end: merci et joyeux anniversaire