Decoupling approach revisited

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Challenges and advances in modern variational analysis
Celebrating the 80th birthday of Professor Jean-Paul Penot
Limoges, March 14-15, 2023

Paseky, April 30, 2003

Decoupling approach

$f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$

$$X \ni x \mapsto (f_1 + f_2)(x)$$
$$X \times X \ni (x_1, x_2) \mapsto f_1(x_1) + f_2(x_2)$$

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$$d(x_1, x_2) \quad \text{is small}$$

Outline

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1. Uniform infimum and uniform lower semicontinuity
2. Quasiuniform infimum and quasiuniform lower semicontinuity
3. Uniform infimum over a set
4. Quasiuniform minimum: fuzzy multiplier rules

Uniform infimum

X is a metric space, $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$, $U \subset X$

$$\inf_U (f_1 + f_2)$$

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Uniform infimum:

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Decoupled infimum (Borwein & Zhu, 2005)
Stabilized infimum (Penot, 2013)

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$$\Lambda_U(f_1, f_2) \le \Lambda_U^\circ(f_1, f_2) \le \inf_U (f_1 + f_2)$$

Uniform lower semicontinuity

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2. (f_1, f_2) is uniformly lsc near a point $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ if it is uniformly lsc on $\overline{B}_\delta(\bar{x})$ for all sufficiently small $\delta > 0$

Firm uniform lower semicontinuity

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$$\Theta_U^\circ(f_1, f_2) := \limsup_{\substack{d(x_1, x_2) \to 0 \\ x_1 \in \mathrm{dom}\, f_1 \cap U,\, x_2 \in \mathrm{dom}\, f_2 \cap U}} (f_1 \diamondsuit f_2)_U(x_1, x_2)$$

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$$\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \cap U \ne \varnothing$$

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(ULC) property (Borwein & Ioffe, 1996)

$\Theta^\circ_U(f_1,f_2) = 0 \quad \Leftrightarrow \quad \forall \{x_{1k}\} \subset \mathrm{dom}\, f_1 \cap U, \{x_{2k}\} \subset \mathrm{dom}\, f_2 \cap U$ with $d(x_{1k},x_{2k}) \to 0$, $\exists \{x_k\} \subset U$ s.t. $d(x_k,x_{1k}) \to 0$ and

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$$\limsup_{k\to+\infty} \left((f_1 + f_2)(x_k) - f_1(x_{1k}) - f_2(x_{2k})\right) \le 0$$

Sequential uniform lower semicontinuity (Borwein & Zhu, 2005); *coherent family* (Penot, 2013)

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$$\Lambda_U^\dagger(f_1, f_2) := \inf_{V \in EI(U)} \liminf_{x_1, x_2 \in V;\, d(x_1,x_2) \to 0} (f_1(x_1) + f_2(x_2))$$

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Essentially interior subset: $V \in EI(U) \;\Leftrightarrow\; \exists\rho>0$ s.t. $B_\rho(V)\subset U$

Uniform lower semicontinuity

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$\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap U \neq \varnothing$

Definition

1. (f_1, f_2) is uniformly lsc on U if $\inf_U (f_1 + f_2) \leq \Lambda^\circ_U(f_1, f_2)$
2. (f_1, f_2) is quasiuniformly lsc on U if $\inf_U (f_1 + f_2) \leq \Lambda^\dagger_U(f_1, f_2)$
3. (f_1, f_2) is firmly uniformly lsc on U if $\Theta^\circ_U(f_1, f_2) = 0$
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$$(3) \Rightarrow (4) \Rightarrow (2) \quad \text{and} \quad (3) \Rightarrow (1) \Rightarrow (2)$$

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3. (f_1, f_2) is firmly uniformly lsc on U if $\Theta_U^{\circ}(f_1, f_2) = 0$
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Proposition

Suppose (f_1, f_2) is firmly uniformly (resp., firmly quasiuniformly) lsc, and $g : X \to \mathbb{R}$ is uniformly continuous on U. Then $(f_1, f_2 + g)$ is firmly uniformly (resp., firmly quasiuniformly) lsc on U.

Uniform lower semicontinuity: sufficient conditions

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Proposition

Suppose f_1 and f_2 are lsc on U, and $\inf_U f_2 > -\infty$

1. (Lassonde, 2001; Penot, 2013) (f_1, f_2) *is uniformly lsc on U if U has compact intersections with sublevel sets of f_1*

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1. (Lassonde, 2001; Penot, 2013) (f_1, f_2) *is uniformly lsc on U if U has compact intersections with sublevel sets of f_1*
2. (f_1, f_2) *is quasiuniformly lsc on U if U has relatively compact intersections with sublevel sets of f_1*

Uniform lower semicontinuity: sufficient conditions

X is a normed space, $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$, $U \subset X$, $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap U \neq \emptyset$

Proposition

Suppose f_1 and f_2 are weakly sequentially lsc on U, and $\inf_U f_2 > -\infty$

1. *(f_1, f_2) is uniformly lsc on U if U has weakly sequentially compact intersections with sublevel sets of f_1*
2. *(f_1, f_2) is quasiuniformly lsc on U if U has relatively weakly sequentially compact intersections with sublevel sets of f_1*

Firm uniform lower semicontinuity: sufficient conditions

X is a metric space, $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$, $U \subset X$,
$\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \cap U \neq \varnothing$

Proposition

(f_1, f_2) is firmly uniformly lsc on U provided that one of the following conditions is satisfied:

1. *there is a $c \in \mathbb{R}$ such that $f_2(x) = c$ for all $x \in \mathrm{dom}\, f_1 \cap U$, and $f_2(x) \geq c$ for all $x \in U \setminus \mathrm{dom}\, f_1$*

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2. *$\mathrm{dom}\, f_2 \cap U = \{\bar{x}\}$, $\bar{x} \in \mathrm{dom}\, f_1$, and f_1 is lsc at $\bar{x}$*

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2. $\mathrm{dom}\, f_2 \cap U = \{\bar{x}\}$, *$\bar{x} \in \mathrm{dom}\, f_1$, and f_1 is lsc at $\bar{x}$*
3. *f_2 is uniformly continuous on U*

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Proposition

(f_1, f_2) is firmly uniformly lsc on U provided that one of the following conditions is satisfied:

1. *there is a $c \in \mathbb{R}$ such that $f_2(x) = c$ for all $x \in \operatorname{dom} f_1 \cap U$, and $f_2(x) \geq c$ for all $x \in U \setminus \operatorname{dom} f_1$*
2. *$\operatorname{dom} f_2 \cap U = \{\bar{x}\}$, $\bar{x} \in \operatorname{dom} f_1$, and f_1 is lsc at $\bar{x}$*
3. *f_2 is uniformly continuous on U*

(3): Borwein & Zhu, 1996; Lasssonde, 2001; Borwein & Zhu, 2005

Outline

Uniform infimum over a set

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$$\Lambda^\dagger_U(f, i_\Omega) = \inf_{V \in EI(U)} \liminf_{x \in V,\, d(x,\Omega) \to 0} f(x)$$

$$\Theta^\circ_U(f, i_\Omega) = \limsup_{\substack{d(x,\Omega \cap U) \to 0 \\ x \in \operatorname{dom} f \cap U}} \inf_{u \in \Omega \cap U} \max\{d(u,x), f(u) - f(x)\}$$

$$\Theta^\dagger_U(f, i_\Omega) = \sup_{V \in EI(U)} \limsup_{\substack{d(x,\Omega) \to 0 \\ x \in \operatorname{dom} f \cap V}} \inf_{u \in \Omega \cap U} \max\{d(u,x), f(u) - f(x)\}$$

Relative uniform lower semicontinuity

X is a metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$, $\Omega, U \subset X$, $\operatorname{dom} f \cap \Omega \cap U \neq \emptyset$

Definition

1. f is uniformly lsc relative to Ω on U if $\inf_{u \in \Omega \cap U} f(u) \le \Lambda^{\circ}_U(f, i_\Omega)$
2. f is quasiuniformly lsc relative to Ω on U if $\inf_{u \in \Omega \cap U} f(u) \le \Lambda^{\dagger}_U(f, i_\Omega)$
3. f is firmly uniformly lsc relative to Ω on U if $\Theta^{\circ}_U(f, i_\Omega) = 0$
4. f is firmly quasiuniformly lsc relative to Ω on U if $\Theta^{\dagger}_U(f, i_\Omega) = 0$
5. f is uniformly/quasiuniformly/firmly uniformly/firmly quasiuniformly lsc relative to Ω near $\bar{x} \in \operatorname{dom} f \cap \Omega$ if it is uniformly/quasiuniformly/firmly uniformly/firmly quasiuniformly lsc relative to Ω on $\overline{B}_\delta(\bar{x})$ for all sufficiently small $\delta > 0$

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(2) and (4): Kruger & Mehlitz, 2022

Outline

Uniform minimum

X is a metric space, $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$, $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$

Definition

1. (Lassonde, 2001) $\bar{x}$ is a local uniform minimum of $f_1 + f_2$ if

$$(f_1 + f_2)(\bar{x}) = \Lambda^{\circ}_{B_\delta(\bar{x})}(f_1, f_2) \quad \text{for some} \quad \delta > 0$$

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Proposition

Suppose (f_1, f_2) is uniformly (quasiuniformly) lsc near $\bar{x}$.
$\bar{x}$ is a local uniform (quasiuniform) minimum of $f_1 + f_2$ $\Leftrightarrow$
$\bar{x}$ is a local minimum of $f_1 + f_2$.

Quasiuniform minimum: fuzzy multiplier rules

X is a Banach space, $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ lsc, $\bar{x} \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$

Theorem

Let $\bar{x}$ be a local quasiuniform minimum of $f_1 + f_2$. Then $\forall \varepsilon > 0$, $\exists x_1, x_2 \in X$ s.t. $\|x_i - \bar{x}\| < \varepsilon$, $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$ $(i = 1, 2)$, $f_1(x_1) + f_2(x_2) \leq (f_1 + f_2)(\bar{x})$, and

$$d(0, \partial^C f_1(x_1) + \partial^C f_2(x_2)) < \varepsilon.$$

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If X is Asplund, then $\forall \varepsilon > 0$, $\exists x_1, x_2 \in X$ s.t. $\|x_i - \bar{x}\| < \varepsilon$, $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$ $(i = 1, 2)$, and

$$d(0, \partial^F f_1(x_1) + \partial^F f_2(x_2)) < \varepsilon.$$

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Fuzzy multiplier rules under (firm) uniform lower semicontinuity:
Borwein & Ioffe, 1996; Borwein & Zhu, 1996; Borwein & Zhu, 2005

Fuzzy sum rule

X is a Asplund space, $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ lsc,
$\bar{x} \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$

Theorem

Suppose that one of the following conditions is satisfied:

(a) (f_1, f_2) *is firmly quasiuniformly lsc near* $\bar{x}$;

(b) X *is reflexive, and* f_1 *and* f_2 *are weakly sequentially lsc;*

(c) $\dim X < +\infty$.

Then $\forall x^* \in \partial^F(f_1 + f_2)(\bar{x})$ *and* $\varepsilon > 0$, $\exists x_1, x_2 \in X$ *s.t.* $\|x_i - \bar{x}\| < \varepsilon$, $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$ $(i = 1, 2)$, *and*

$$d(x^*, \partial^F f_1(x_1) + \partial^F f_2(x_2)) < \varepsilon.$$

Fuzzy intersection rule

X is a reflexive Banach space, $\Omega_1, \Omega_2 \subset X$, $\bar{x} \in \Omega_1 \cap \Omega_2$

Corollary

Let Ω_1, Ω_2 be weakly sequentially closed. Then, $\forall x^* \in N^F_{\Omega_1 \cap \Omega_2}(\bar{x})$ *and* $\forall \varepsilon > 0$, $\exists x_1 \in \Omega_1 \cap B_\varepsilon(\bar{x})$ *and* $x_2 \in \Omega_2 \cap B_\varepsilon(\bar{x})$ *s.t.*

$$d(x^*, N^F_{\Omega_1}(x_1) + N^F_{\Omega_2}(x_2)) < \varepsilon.$$

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Prague, May 6, 2003

Happy Birthday Jean-Paul!