Geometric characterizations of some differentiability concepts of sets

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Abstract. In this talk, we will investigate geometrical characterizations of two concepts of differentiabilty:

(1) Clarke regularity of subanalytic sets;
(2) Strictly Hadamard differentiabilty of epi-Lipschitzian sets.

In a finite dimensional space X, we will show that for a closed subanalytic subset S, the Clarke tangential regularity of S at $\bar{x} \in S$ is equivalent to the coincidence of the Clarke's tangent cone to S at $\bar{x}$ with the set

$$\mathcal{L}(S, \bar{x}) := \left\{ \dot{c}_+(0) \in X : \, c : [0,1] \longrightarrow S \; \textit{ is Lipschitz}, \, c(0) = \bar{x} \right\}.$$

In a Banach space, we will show that for an epi-Lipschitzian set S at $\bar{x}$ in the boundary of S, the following assertions are equivalent:

- S is strictly Hadamard differentiable at $\bar{x}$;
- the Clarke tangent cone $T(S, \bar{x})$ to S at $\bar{x}$ contains a closed hyperplane;
- the Clarke tangent cone $T(\operatorname{bdry} S, \bar{x})$ to $\operatorname{bdry} S$ at $\bar{x}$ is a closed hyperplane.

Moreover when X is of finite dimension, Y is a Banach space and $g : X \mapsto Y$ is a locally Lipschitz mapping around $\bar{x}$, we show that g is strictly Hadamard differentiable at $\bar{x}$ IFF $T(\operatorname{graph} g, (\bar{x}, g(\bar{x})))$ is isomorphic to X IFF the set-valued mapping $x \rightrightarrows K(\operatorname{graph} g, (x, g(x)))$ is continuous at $\bar{x}$ and $K(\operatorname{graph} g, (\bar{x}, g(\bar{x})))$ is isomorphic to X, where $K(A, a)$ denotes the contingent cone to a set A at $a \in A$.

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